

# Spectrum Analysis for the Vlasov-Poisson-Boltzmann System

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## Abstract

By identifying a norm capturing the effect of the forcing governed by the Poisson equation, we give a detailed spectrum analysis on the linearized Vlasov-Poisson-Boltzmann system around a global Maxwellian. It is shown that the electric field governed by the self-consistent Poisson equation plays a key role in the analysis so that the spectrum structure is genuinely different from the well-known one of the Boltzmann equation. Based on this, we give the optimal time decay rates of solutions to the equilibrium.

**Key words.** Vlasov-Poisson-Boltzmann system, spectrum analysis, optimal time decay rates.

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## 1 Introduction

The Vlasov-Poisson-Boltzmann (VPB) system can be used to describe the motion of the dilute charged particles in plasma or semiconductor devices under the influence of the self-consistent electric field [14]. In the present paper, we consider the Cauchy problem for VPB system for one species in  $\mathbb{R}_x^3 \times \mathbb{R}_v^3 \times \mathbb{R}_+$ :

$$\begin{cases} F_t + v \cdot \nabla_x F + \nabla_x \Phi \cdot \nabla_v F = \mathcal{Q}(F, F), \\ \Delta_x \Phi = \int_{\mathbb{R}^3} F dv - \bar{\rho}, \\ F(x, v, 0) = F_0(x, v), \end{cases} \quad (1.1)$$

where  $F = F(x, v, t)$  is the distribution function, and  $\Phi(x, t)$  denotes the electrostatic potential.  $\bar{\rho} > 0$  is the given doping density and assumed to be a constant. As usual, the operator  $\mathcal{Q}(F, G)$  describing the binary elastic collision between particles takes the form

$$\mathcal{Q}(F, G) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \omega) (F(v'_*)G(v') - F(v_*)G(v)) dv_* d\omega, \quad (1.2)$$

where

$$v' = v - [(v - v_*) \cdot \omega] \omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega] \omega, \quad \omega \in \mathbb{S}^2.$$

For monatomic gas, the collision kernel  $B(|v - v_*|, \omega)$  is a non-negative function of  $|v - v_*|$  and  $|(v - v_*) \cdot \omega|$ :

$$B(|v - v_*|, \omega) = B(|v - v_*|, \cos \theta), \quad \cos \theta = \frac{|(v - v_*) \cdot \omega|}{|v - v_*|}, \quad \theta \in [0, \pi/2].$$

In the following, we consider both the hard sphere model and hard potential with angular cutoff. Precisely, for the hard sphere model,

$$B(|v - v_*|, \omega) = |(v - v_*) \cdot \omega| = |v - v_*| \cos \theta; \quad (1.3)$$

and for the models of the hard potentials with Grad angular cutoff assumption,

$$B(|v - v_*|, \omega) = b(\cos \theta) |v - v_*|^\gamma, \quad 0 \leq \gamma < 1, \quad (1.4)$$

where we assume for simplicity

$$0 \leq b(\cos \theta) \leq C |\cos \theta|.$$

There have been a lot of works on the existence and behavior of solutions to the Vlasov-Poisson-Boltzmann system. The global existence of renormalized solution for large initial data was proved in [15]. The first global existence result on classical solution in torus when the initial data is near a global Maxwellian was established in [8]. And the global existence of classical solution in  $\mathbb{R}^3$  was given [20, 21] in the same setting. The case with general stationary background density function  $\bar{\rho}(x)$  was studied in [3], and the perturbation of vacuum was investigated in [9, 5]

However, in contrast to the works on Boltzmann equation [7, 12, 13, 17, 18, 19], the spectrum of the linearized VPB system has not been given despite of its importance. On the other hand, an interesting phenomenon was shown recently in [3] on the time asymptotic behavior of the solutions which shows that the global classical solution of one species VPB system tends to the equilibrium at  $(1+t)^{-\frac{1}{4}}$  in  $L^2$ -norm. This is slower than the rate for the two species VPB system, that is,  $(1+t)^{-\frac{3}{4}}$ , obtained in [22]. Therefore, it is natural to investigate whether these rates are optimal.

Following the approach on the spectrum analysis on the Boltzmann equation [7], we now consider the VPB system as follow. To begin with, we take  $\bar{\rho} = 1$  for simplicity. The VPB system (1.1) has a stationary solution  $(F_*, \Phi_*) = (M(v), 0)$  with the normalized Maxwellian  $M(v)$  given by

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^3.$$

As usual, define the perturbation  $f(x, v, t)$  of  $F(x, v, t)$  near  $M$  by

$$F = M + \sqrt{M}f,$$

then the VPB system (1.1) for  $F$  is reformulated in terms of  $f$  into

$$\begin{cases} f_t = Bf + G(f), & t > 0, \\ \Delta_x \Phi = \int_{\mathbb{R}^3} f \sqrt{M} dv, \\ f(x, v, 0) = f_0(x, v) = (F_0 - M)M^{-\frac{1}{2}}, \end{cases} \quad (1.5)$$

where the operator  $B$  is defined by

$$Bf = Lf - v \cdot \nabla_x f - v \sqrt{M} \cdot \nabla_x (-\Delta_x)^{-1} \left( \int_{\mathbb{R}^3} f \sqrt{M} dv \right), \quad (1.6)$$

and the nonlinear term  $G$  is given by

$$G =: G_1 + G_2, \quad G_1 = \Gamma(f, f), \quad G_2 = \frac{1}{2}(v \cdot \nabla_x \Phi)f - \nabla_x \Phi \cdot \nabla_v f. \quad (1.7)$$

The linearized collision operator  $Lf$  and the nonlinear term  $\Gamma(f, f)$  in (1.5) are defined by

$$Lf = \frac{1}{\sqrt{M}} [\mathcal{Q}(M, \sqrt{M}f) + \mathcal{Q}(\sqrt{M}f, M)], \quad (1.8)$$

$$\Gamma(f, f) = \frac{1}{\sqrt{M}} \mathcal{Q}(\sqrt{M}f, \sqrt{M}f). \quad (1.9)$$

We have, cf [1],

$$\begin{aligned} (Lf)(v) &= (Kf)(v) - \nu(v)f(v), \\ \nu(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \omega) M_* d\omega dv_*, \\ (Kf)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \omega) (\sqrt{M'_*} f'_* + \sqrt{M'} f'_* - \sqrt{M} f_*) \sqrt{M_*} d\omega dv_* \\ &= \int_{\mathbb{R}^3} k(v, v_*) f(v_*) dv_*, \end{aligned} \quad (1.10)$$

where  $\nu(v)$  is called the collision frequency and  $K$  is a self-adjoint compact operator on  $L^2(\mathbb{R}_v^3)$  with a real symmetric integral kernel  $k(v, v_*)$ . The null space of the operator  $L$ , denoted by  $N_0$ , is a subspace spanned by the orthonormal basis  $\{\chi_j, j = 0, 1, \dots, 4\}$  with

$$\chi_0 = \sqrt{M}, \quad \chi_j = v_j \sqrt{M} \quad (j = 1, 2, 3), \quad \chi_4 = \frac{(|v|^2 - 3)\sqrt{M}}{\sqrt{6}}. \quad (1.11)$$

Let  $\mathbf{P}_0$  be the projection operator from  $L^2(\mathbb{R}_v^3)$  to the subspace  $N_0$  and  $\mathbf{P}_1 = I - \mathbf{P}_0$ , and  $L^2(\mathbb{R}^3)$  be a Hilbert space of complex-value functions  $f(v)$  on  $\mathbb{R}^3$  with the inner product and the norm

$$(f, g) = \int_{\mathbb{R}^3} f(v) \overline{g(v)} dv, \quad \|f\| = \left( \int_{\mathbb{R}^3} |f(v)|^2 dv \right)^{1/2}.$$

From the Boltzmann's H-theorem, the linearized collision operator  $L$  is non-positive, and moreover,  $L$  is locally coercive in the sense that there is a constant  $\mu > 0$  such that

$$(Lf, f) \leq -\mu \|\mathbf{P}_1 f\|^2, \quad f \in D(L), \quad (1.12)$$

where  $D(L)$  is the domain of  $L$  given by

$$D(L) = \{f \in L^2(\mathbb{R}^3) \mid \nu(v)f \in L^2(\mathbb{R}^3)\}.$$

In addition,  $\nu(v)$  satisfies

$$\nu_0(1 + |v|)^\gamma \leq \nu(v) \leq \nu_1(1 + |v|)^\gamma, \quad (1.13)$$

with  $\gamma = 1$  for hard sphere and  $0 \leq \gamma < 1$  for hard potential. Without the loss of generality, we assume in this paper that  $\nu(0) \geq \nu_0 \geq \mu > 0$ .

The solution  $f$  can be decomposed into the fluid part and the non-fluid part as

$$\begin{cases} f = \mathbf{P}_0 f + \mathbf{P}_1 f, \\ \mathbf{P}_0 f = n\chi_0 + \sum_{j=1}^3 m_j \chi_j + q\chi_4, \end{cases} \quad (1.14)$$

where the density  $n$ , the momentum  $m = (m_1, m_2, m_3)$  and the energy  $q$  are defined by

$$(f, \chi_0) = n, \quad (f, \chi_j) = m_j, \quad (f, \chi_4) = q.$$

In this paper, we will show that the electric field force influences the structure of the spectra and resolvent sets of the linearized VPB system (2.1), the low frequency expansions of eigenvalues and the corresponding eigenfunctions. In particular, the spectra of linearized operator contain five eigenvalues in low frequency which locate in three different small neighborhoods centered at points  $\lambda = \pm i, 0$  respectively. Moreover, the low frequency asymptotic expansions of the eigenvalues in the small neighborhood of the points  $\lambda = \pm i$  have different structure up to the second order from those of the Boltzmann equation, so do the asymptotic expansions of the corresponding eigenfunctions for which higher order expansions are required in order to determine the coefficients at lower order with respect to the frequency (see Lemma 2.6, Theorem 2.11 and Remark 2.12 for details). This phenomenon is mainly due to the fact that the macroscopic velocity vector field is affected by the electric field. In the absence of the electric field, the two small neighborhoods centered at the points  $\lambda = \pm i$  merge formally with the one centered at the origin (see Remark 2.10 for details).

With the help of the spectrum analysis on the linearized VPB system (2.2), we are able to represent the global solution in terms of the corresponding semigroup, and in particular, to decompose the semigroup by using the resolvent and spectral representation of its generator with respect to the frequency. Therefore, we can show that the global solutions to the linearized VPB system (2.1) decay to zero at the optimal time decay rate  $(1+t)^{-1/4}$  in  $L^2$ -norm (see Theorem 3.6), which is slower than  $(1+t)^{-3/4}$  for the linearized Boltzmann equation. Furthermore, we prove that the first and third moments (macroscopic density and energy) decay at the same optimal rate  $(1+t)^{-3/4}$  in  $L^2$ -norm like the Boltzmann equation. However, the second moment (macroscopic momentum) decays at the optimal rate  $(1+t)^{-1/4}$ , and the microscopic part of the solution decays at the optimal rate  $(1+t)^{-3/4}$ , which are slower than the corresponding optimal rates for the Boltzmann equation.

Finally, we will prove that the global solution to the Cauchy problem for the original VPB system (1.5) tends to the global Maxwellian at the optimal time-decay rate  $(1+t)^{-1/4}$  in  $L^2$ -norm (see Theorem 4.2), which is slower than the one for the Boltzmann equation. More precisely, we show that the first moment (macroscopic density) of the global solution converges to its equilibrium state at the same optimal rate  $(1+t)^{-3/4}$  as the Boltzmann equation, but the second moment (macroscopic momentum) decays at the optimal rate  $(1+t)^{-1/4}$ . And the microscopic part of the global solution decays at the optimal rate  $(1+t)^{-3/4}$ . Some comparison with the Navier-Stokes-Poisson system will also be given.

The rest of this paper will be organized as follows. In Section 2, we study the spectrum and resolvent of the linear operator  $\hat{B}(\xi)$  related to the linearized VPB system and establish asymptotic expansions of the eigenvalues and eigenfunctions at low frequency. In Section 3, we decompose the semigroup  $e^{t\hat{B}(\xi)}$  generated by the linear operator  $\hat{B}(\xi)$  with respect to the low frequency and high frequency, and then establish the optimal time decay rates of the global solution to the linearized VPB system in terms of the semigroup  $e^{tB}$ . In Section 4, we prove the optimal time decay rates of the global solution to the original nonlinear VPB system. Finally, we present in Section 5 some additional discussion about the influence of the electric field on the time

decay rate of the nonlinear VPB system in terms of some detailed analysis on the compressible Navier-Stokes-Poisson equations for the macroscopic density, momentum and temperature coupled with the equation for the microscopic component. Some well-known results on the semigroup theory are recalled in Section 6 for the easy reference of the readers.

**Notations:**  $\hat{f}(\xi, v) = \mathcal{F}f(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x, v) e^{-ix \cdot \xi} dx$  denotes the Fourier transform of  $f = f(x, v)$ . Throughout this paper,  $\mathbb{R}_{x,v}^3 =: \mathbb{R}_x^3 \times \mathbb{R}_v^3$ ,  $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ ,  $\xi^\alpha =: \xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3}$  with  $\xi = (\xi_1, \xi_2, \xi_3)$  and  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ . We denote by  $\|\cdot\|_{L_{x,v}^2}$  and  $\|\cdot\|_{L_{\xi,v}^2}$  the norm of the function spaces  $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$  and  $L^2(\mathbb{R}_\xi^3 \times \mathbb{R}_v^3)$  respectively, denote by  $\|\cdot\|_{L_v^2(H_x^N)}$  the norm of the function space  $L^2(\mathbb{R}_v^3, H^N(\mathbb{R}_x^3))$  with  $N \geq 1$  integer, and denote by  $\|\cdot\|_{L_x^2}$ ,  $\|\cdot\|_{L_\xi^2}$  and  $\|\cdot\|_{L_v^2}$  the norms of the function spaces  $L^2(\mathbb{R}_x^3)$ ,  $L^2(\mathbb{R}_\xi^3)$  and  $L^2(\mathbb{R}_v^3)$  respectively. Moreover,  $C > 0$ ,  $C_1 > 0$ ,  $C_2 > 0$  and  $\beta > 0$  denote some generic constants.

## 2 Linear VPB and spectral analysis

In this section, we study the spectrum and resolvent set of the linear operator related to  $B$  in order to obtain the optimal decay rate of the global solution to IVP (1.5). To this end, we consider the linear Vlasov-Poisson-Boltzmann equation in this section

$$\begin{cases} f_t = Bf, & t > 0, \\ f(0, x, v) = f_0(x, v), & (x, v) \in \mathbb{R}_x^3 \times \mathbb{R}_v^3, \end{cases} \quad (2.1)$$

with the operator  $B$  defined by (1.6).

### 2.1 Spectrum and resolvent

We take the Fourier transform in (2.1) with respect to  $x$

$$\begin{cases} \hat{f}_t = \hat{B}(\xi) \hat{f}, & t > 0, \\ \hat{f}(0, \xi, v) = \hat{f}_0(\xi, v), & (\xi, v) \in \mathbb{R}_\xi^3 \times \mathbb{R}_v^3, \end{cases} \quad (2.2)$$

where the operator  $\hat{B}(\xi)$  is defined for  $\xi \neq 0$  by

$$\hat{B}(\xi) \hat{f} = L \hat{f} - i(v \cdot \xi) \hat{f} - \frac{i(v \cdot \xi)}{|\xi|^2} \sqrt{M} \int_{\mathbb{R}^3} \hat{f} \sqrt{M} dv.$$

Set

$$\hat{B}(\xi) = L - i(v \cdot \xi) - \frac{i(v \cdot \xi)}{|\xi|^2} P_d, \quad (2.3)$$

with

$$P_d f = \sqrt{M} \int_{\mathbb{R}^3} f \sqrt{M} dv, \quad f \in L^2(\mathbb{R}^3). \quad (2.4)$$

For each  $\xi \neq 0$ , it is obvious that the operator  $\frac{i(v \cdot \xi)}{|\xi|^2} P_d$  is compact on  $L^2(\mathbb{R}^3)$ .

Introduce the weighted Hilbert space  $L_\xi^2(\mathbb{R}_v^3)$  for  $\xi \neq 0$  as

$$L_\xi^2(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}_v^3) \mid \|f\|_\xi = \sqrt{(f, f)_\xi} < \infty\},$$

equipped with the inner product

$$(f, g)_\xi = (f, g) + \frac{1}{|\xi|^2} (P_d f, P_d g).$$

Since  $P_d$  is a self-adjoint operator and satisfies  $(P_d f, P_d g) = (P_d f, g) = (f, P_d g)$ , we have

$$(f, g)_\xi = (f, g + \frac{1}{|\xi|^2} P_d g) = (f + \frac{1}{|\xi|^2} P_d f, g). \quad (2.5)$$

We can regard  $\hat{B}(\xi)$  as a linear operator from the space  $L_\xi^2(\mathbb{R}^3)$  to itself because

$$\|f\|^2 \leq \|f\|_\xi^2 \leq (1 + |\xi|^{-2})\|f\|^2, \quad \xi \neq 0.$$

In particular, we show

**Lemma 2.1.** *The operator  $\hat{B}(\xi)$  generates a strongly continuous contraction semigroup on  $L_\xi^2(\mathbb{R}_v^3)$  satisfying*

$$\|e^{t\hat{B}(\xi)}f\|_\xi \leq \|f\|_\xi, \quad \text{for } t > 0, f \in L_\xi^2(\mathbb{R}_v^3). \quad (2.6)$$

*Proof.* First we show that both  $\hat{B}(\xi)$  and  $\hat{B}(\xi)^*$  are dissipative operators on  $L_\xi^2(\mathbb{R}_v^3)$ . By (2.5), we obtain for any  $f, g \in L_\xi^2(\mathbb{R}_v^3) \cap D(\hat{B}(\xi))$  that  $(\hat{B}(\xi)f, g)_\xi = (f, \hat{B}(\xi)^*g)_\xi$  because

$$(\hat{B}(\xi)f, g)_\xi = (\hat{B}(\xi)f, g + \frac{1}{|\xi|^2}P_d g) = (f, (L + i(v \cdot \xi) + \frac{i(v \cdot \xi)}{|\xi|^2}P_d)g)_\xi = (f, \hat{B}(\xi)^*g)_\xi, \quad (2.7)$$

with  $\hat{B}(\xi)^* = \hat{B}(-\xi) = L + i(v \cdot \xi) + \frac{i(v \cdot \xi)}{|\xi|^2}P_d$ . Direct computation gives rise to the dissipation of both  $\hat{B}(\xi)$  and  $\hat{B}(\xi)^*$ , namely,  $\text{Re}(\hat{B}(\xi)f, f)_\xi = \text{Re}(\hat{B}(\xi)^*f, f)_\xi = (Lf, f) \leq 0$ . Since  $\hat{B}(\xi)$  is a densely defined closed operator, it follows from Lemma 6.2 that the operator  $\hat{B}(\xi)$  generates a  $C_0$ -contraction semigroup on  $L_\xi^2(\mathbb{R}_v^3)$ .  $\square$

Denote by  $\rho(\hat{B}(\xi))$  the resolvent set and by  $\sigma(\hat{B}(\xi))$  the spectrum set of  $\hat{B}(\xi)$ . We have

**Lemma 2.2.** *For each  $\xi \neq 0$ , the spectrum set  $\sigma(\hat{B}(\xi))$  of the operator  $\hat{B}(\xi)$  on the domain  $\text{Re}\lambda \geq -\nu_0 + \delta$  for any constant  $\delta > 0$  consists of isolated eigenvalues  $\Sigma =: \{\lambda_j(\xi)\}$  with  $\text{Re}\lambda_j(\xi) < 0$ .*

*Proof.* Define

$$c(\xi) = -\nu(v) - i(v \cdot \xi). \quad (2.8)$$

It's obvious that  $\lambda - c(\xi)$  is invertible for  $\text{Re}\lambda > -\nu_0$ . Since  $K$  and  $\frac{i(v \cdot \xi)}{|\xi|^2}P_d$  are compact operators on  $L_\xi^2(\mathbb{R}_v^3)$  for any fixed  $\xi \neq 0$ ,  $\hat{B}(\xi)$  is a compact perturbation of  $c(\xi)$ , and so, thanks to Theorem 5.35 in p.244 of [10],  $\hat{B}(\xi)$  and  $c(\xi)$  have the same essential spectrum. Thus the spectrum of  $\hat{B}(\xi)$  in the domain  $\text{Re}\lambda > -\nu_0$  consists of discrete eigenvalues  $\lambda_j(\xi)$  with possible accumulation points only on the line  $\text{Re}\lambda = -\nu_0$ .

We claim that for any discrete eigenvalue  $\lambda(\xi)$  of  $\hat{B}(\xi)$  on the domain  $\text{Re}\lambda \geq -\nu_0 + \delta$  for any constant  $\delta > 0$ , it holds that  $\text{Re}\lambda(\xi) < 0$  for  $\xi \neq 0$ . Indeed, set  $\xi = s\omega$  and let  $h$  be the eigenfunction corresponding to the eigenvalue  $\lambda$  so that

$$\lambda h = Lh - is(v \cdot \omega) \left( h + \frac{1}{s^2}P_d h \right). \quad (2.9)$$

Taking the inner product between (2.9) and  $h + \frac{1}{s^2}P_d h$  and choosing the real part, we have

$$(Lh, h) = \text{Re}\lambda \left( \|h\|^2 + \frac{1}{s^2}\|P_d h\|^2 \right),$$

which together with (1.12) implies  $\text{Re}\lambda \leq 0$ .

Furthermore, if there exists an eigenvalue  $\lambda$  with  $\text{Re}\lambda = 0$ , then it follows from the above that  $(Lh, h) = 0$ , namely, the corresponding eigenfunction  $h$  belongs to the nullspace of the operator  $L$ , i.e.,  $h \in N_0$  and

$$-is(v \cdot \omega) \left( h + \frac{1}{s^2}P_d h \right) = \lambda h,$$

which, after projected into the null space  $N_0$  and its orthogonal complement  $N_0^\perp$ , leads to

$$\mathbf{P}_0(v \cdot \omega) \left( sh + \frac{1}{s}P_d h \right) = i\lambda h, \quad (2.10)$$

$$\mathbf{P}_1(v \cdot \omega)h = 0. \quad (2.11)$$

On the other hand, the function  $h \in N_0$  can be represented in terms of the five basis of  $N_0$  as

$$h = C_0\sqrt{M} + \sum_{j=1}^3 C_j v_j \sqrt{M} + C_4 \frac{(|v|^2-3)}{\sqrt{6}} \sqrt{M},$$

we can deduce after direct computation that

$$\begin{aligned} \mathbf{P}_0(v \cdot \omega)h &= (C_0 + \sqrt{\frac{2}{3}}C_4)(v \cdot \omega)\sqrt{M} + \frac{1}{3} \sum_{j=1}^3 C_j \omega_j |v|^2 \sqrt{M}, \\ \mathbf{P}_1(v \cdot \omega)h &= \sum_{i,j=1}^3 C_i \omega_j \left( v_i v_j - \delta_{ij} \frac{|v|^2}{3} \right) \sqrt{M} + C_4 (v \cdot \omega) \left( \frac{|v|^2-3}{\sqrt{6}} - \sqrt{\frac{2}{3}} \right) \sqrt{M}. \end{aligned}$$

This together with (2.11) imply that  $C_i = 0$  for  $i = 1, 2, 3, 4$ , namely,  $h = C_0\sqrt{M}$ . Substituting it into (2.10), we can obtain

$$(v \cdot \omega) \left( s + \frac{1}{s} \right) C_0 \sqrt{M} = i\lambda C_0 \sqrt{M},$$

which implies  $C_0 = 0$ . Therefore, we conclude that  $h \equiv 0$ . This is a contradiction and thus it holds  $\text{Re}\lambda < 0$  for all discrete eigenvalues  $\lambda \in \sigma(\hat{B}(\xi))$ .  $\square$

Now denote by  $T$  a linear operator on  $L^2(\mathbb{R}_v^3)$  or  $L_\xi^2(\mathbb{R}_v^3)$ , and we define the corresponding norms of  $T$  by

$$\|T\| = \sup_{\|f\|=1} \|Tf\|, \quad \|T\|_\xi = \sup_{\|f\|_\xi=1} \|Tf\|_\xi.$$

One can verify that  $\frac{\|Tf\|}{(1+|\xi|^{-2})\|f\|} \leq \frac{\|Tf\|_\xi}{\|f\|_\xi} \leq \frac{(1+|\xi|^{-2})\|Tf\|}{\|f\|}$ , which implies

$$(1 + |\xi|^{-2})^{-1} \|T\| \leq \|T\|_\xi \leq (1 + |\xi|^{-2}) \|T\|. \quad (2.12)$$

We will make use of the following decomposition associated with the operator  $\hat{B}(\xi)$  for  $|\xi| > 0$

$$\begin{aligned} \lambda - \hat{B}(\xi) &= \lambda - c(\xi) - K + \frac{i(v \cdot \xi)}{|\xi|^2} P_d \\ &= (I - K(\lambda - c(\xi))^{-1} + \frac{i(v \cdot \xi)}{|\xi|^2} P_d (\lambda - c(\xi))^{-1})(\lambda - c(\xi)), \end{aligned} \quad (2.13)$$

and estimate the right hand terms of (2.13) as follows.

**Lemma 2.3.** *There exists a constant  $C > 0$  so that it holds:*

1. *For any  $\delta > 0$ , we have*

$$\sup_{x \geq -\nu_0 + \delta, y \in \mathbb{R}} \|K(x + iy - c(\xi))^{-1}\| \leq C\delta^{-11/13}(1 + |\xi|)^{-2/13}. \quad (2.14)$$

2. *For any  $\delta > 0$ ,  $r_0 > 0$ , there is a constant  $y_0 = (2r_0)^{5/3}\delta^{-2/3} > 0$  such that if  $|y| \geq y_0$ , we have*

$$\sup_{x \geq -\nu_0 + \delta, |\xi| \leq r_0} \|K(x + iy - c(\xi))^{-1}\| \leq C\delta^{-3/5}(1 + |y|)^{-2/5}. \quad (2.15)$$

3. *For any  $\delta > 0$ ,  $r_0 > 0$ , we have*

$$\sup_{x \geq -\nu_0 + \delta, y \in \mathbb{R}} \|(v \cdot \xi)|\xi|^{-2} P_d(x + iy - c(\xi))^{-1}\| \leq C\delta^{-1}|\xi|^{-1}, \quad (2.16)$$

$$\sup_{x \geq -\nu_0 + \delta, |\xi| \geq r_0} \|(v \cdot \xi)|\xi|^{-2} P_d(x + iy - c(\xi))^{-1}\| \leq C(r_0^{-1} + 1)(\delta^{-1} + 1)|y|^{-1}. \quad (2.17)$$

*Proof.* The proof of (2.14) and (2.15) is the same as the one in Lemma 2.2.6 in [19] so that we omit its detail.

(2.16) can be obtained by the fact that  $\|(v \cdot \xi)|\xi|^{-2}P_d\| \leq C|\xi|^{-1}$ , and  $\|(x+iy-c(\xi))^{-1}\| \leq \delta^{-1}$  for  $x \geq -\nu_0 + \delta$ . And (2.17) follows from the fact that  $(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1} = \frac{1}{\lambda}(v \cdot \xi)|\xi|^{-2}P_d + \frac{1}{\lambda}(v \cdot \xi)|\xi|^{-2}P_dc(\xi)(\lambda - c(\xi))^{-1}$ , and  $\|(v \cdot \xi)|\xi|^{-2}P_dc(\xi)\| \leq C(r_0^{-1} + 1)$  for  $|\xi| \geq r_0$ .  $\square$

With the help of Lemma 2.3, we can investigate the spectral gap of the operator  $\hat{B}(\xi)$  for high frequency.

**Lemma 2.4** (Spectral gap). *Let  $\lambda(\xi) \in \sigma(\hat{B}(\xi))$  be any eigenvalue of  $\hat{B}(\xi)$  in the domain  $\text{Re}\lambda \geq -\nu_0 + \delta$  with  $\delta > 0$  being a constant. Then, for any  $r_0 > 0$ , there exists  $\alpha(r_0) > 0$  so that  $\text{Re}\lambda(\xi) \leq -\alpha(r_0)$  for all  $|\xi| \geq r_0$ .*

*Proof.* We first show that  $\sup_{|\xi| \geq r_0} |\text{Im}\lambda(\xi)| < +\infty$  for any  $\lambda(\xi) \in \sigma(\hat{B}(\xi))$  with  $\text{Re}\lambda \geq -\nu_0 + \delta$ . Indeed, by (2.14), (2.16) and (2.12), there exists  $r_1 = r_1(\delta) > 0$  large enough so that  $\text{Re}\lambda \geq -\nu_0 + \delta$  and  $|\xi| \geq r_1$ ,

$$\|K(\lambda - c(\xi))^{-1}\|_\xi \leq \frac{1}{4}, \quad \|(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}\|_\xi \leq \frac{1}{4}. \quad (2.18)$$

This implies that the operator  $I + K(\lambda - c(\xi))^{-1} + i(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}$  is invertible on  $L_\xi^2(\mathbb{R}_v^3)$ , which together with (2.13) yield that  $(\lambda - \hat{B}(\xi))$  is also invertible on  $L_\xi^2(\mathbb{R}_v^3)$  for  $\text{Re}\lambda \geq -\nu_0 + \delta$  and  $|\xi| \geq r_1$  and it satisfies

$$(\lambda - \hat{B}(\xi))^{-1} = (\lambda - c(\xi))^{-1}(I - K(\lambda - c(\xi))^{-1} + \frac{i(v \cdot \xi)}{|\xi|^2}P_d(\lambda - c(\xi))^{-1})^{-1}, \quad (2.19)$$

namely,  $\{\lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta\} \subset \rho(\hat{B}(\xi))$  for  $|\xi| \geq r_1$ .

As for  $r_0 \leq |\xi| \leq r_1$ , by (2.15) and (2.17) there is a constant  $\zeta = \zeta(r_0, r_1, \delta) > 0$  so that (2.18) still holds for  $|\text{Im}\lambda| > \zeta$ . This also implies the invertibility of  $(\lambda - \hat{B}(\xi))$ , namely, it holds  $\{\lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta, |\text{Im}\lambda| > \zeta\} \subset \rho(\hat{B}(\xi))$  for  $r_0 \leq |\xi| \leq r_1$ . Thus, we conclude

$$\sigma(\hat{B}(\xi)) \cap \{\lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta\} \subset \{\lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta, |\text{Im}\lambda| \leq \zeta\}, \quad |\xi| \geq r_0. \quad (2.20)$$

Next, we prove that  $\sup_{|\xi| \geq r_0} \text{Re}\lambda(\xi) < 0$ . Based on the above argument, it is sufficient to prove that  $\sup_{r_0 \leq |\xi| \leq r_1} \text{Re}\lambda(\xi) < 0$ . If it does not hold, namely for some given  $r_0 > 0$ , there exist a sequence of  $(\xi_n, \lambda_n, f_n)$  satisfying  $|\xi_n| \in [r_0, r_1]$ ,  $f_n \in L^2(\mathbb{R}^3)$  with  $\|f_n\| = 1$ , and  $\lambda_n \in \sigma(\hat{B}(\xi_n))$  so that

$$(L - i(v \cdot \xi_n))f_n - \frac{i(v \cdot \xi_n)}{|\xi_n|^2}P_df_n = \lambda_n f_n, \quad \text{Re}\lambda_n \rightarrow 0, \quad n \rightarrow \infty.$$

The above equation can be rewritten as  $(\lambda_n + \nu + i(v \cdot \xi_n))f_n = Kf_n - \frac{i(v \cdot \xi_n)}{|\xi_n|^2}P_df_n$ . Since  $K$  is a compact operator on  $L^2(\mathbb{R}^3)$ , there exists a subsequence  $f_{n_j}$  of  $f_n$  and  $g_1 \in L^2(\mathbb{R}^3)$  such that

$$Kf_{n_j} \rightarrow g_1, \quad \text{as } j \rightarrow \infty.$$

Due to the fact that  $|\xi_n| \in [r_0, r_1]$ ,  $P_df_n = C_0^n \sqrt{M}$  with  $|C_0^n| \leq 1$ , there exists a subsequence of (still denoted by)  $(\xi_{n_j}, f_{n_j})$ , and  $(\xi_0, C_0)$  with  $|\xi_0| \in [r_0, r_1]$  and  $|C_0| \leq 1$  such that

$$i(v \cdot \xi_{n_j})|\xi_{n_j}|^{-2}P_df_{n_j} \rightarrow g_2 =: i(v \cdot \xi_0)|\xi_0|^{-2}C_0\sqrt{M}, \quad \text{as } j \rightarrow \infty.$$

Since  $|\text{Im}\lambda_n| \leq \zeta$  and  $\text{Re}\lambda_n \rightarrow 0$ , we can extract a subsequence of (still denoted by)  $\lambda_{n_j}$  such that  $\lambda_{n_j} \rightarrow \lambda_0$  with  $\text{Re}\lambda_0 = 0$ . Noting that  $|\lambda_n + \nu + i(v \cdot \xi_n)| \geq \delta$ , we have

$$\lim_{j \rightarrow \infty} f_{n_j} = \lim_{j \rightarrow \infty} \frac{g_1 - g_2}{\lambda_{n_j} + \nu + i(v \cdot \xi_{n_j})} = \frac{g_1 - g_2}{\lambda_0 + \nu + i(v \cdot \xi_0)} := f_0 \quad \text{in } L^2(\mathbb{R}^3),$$

and hence  $Kf_0 = g_1, i(v \cdot \xi_0)|\xi_0|^{-2}P_df = g_2$ . It follows that  $B(\xi_0)f_0 = \lambda_0 f_0$  and  $\lambda_0$  is an eigenvalue of  $B(\xi_0)$  with  $\text{Re}\lambda_0 = 0$ , which contradicts the fact  $\text{Re}\lambda(\xi) < 0$  for  $\xi \neq 0$  established by Lemma 2.2. The proof the lemma is then completed.  $\square$



Then, we investigate the spectrum and resolvent sets of  $\hat{B}(\xi)$  at low frequency. To this end, we decompose  $\lambda - \hat{B}(\xi)$  as follows

$$\lambda - \hat{B}(\xi) = \lambda \mathbf{P}_0 - A(\xi) + \lambda \mathbf{P}_1 - Q(\xi) + i\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1 + i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_0, \quad (2.21)$$

with the operators  $A(\xi)$  and  $Q(\xi)$  defined by

$$A(\xi) =: -i\mathbf{P}_0(v \cdot \xi)\mathbf{P}_0 - \frac{i(v \cdot \xi)}{|\xi|^2}P_d, \quad Q(\xi) =: L - i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1. \quad (2.22)$$

It is easy to verify that  $A(\xi)$  is a linear operator from the null space  $N_0$  to itself, and can be represented in the basis of  $N_0$  as

$$A(\xi) = \begin{pmatrix} 0 & -i\xi^T & 0 \\ -i\xi(1 + \frac{1}{|\xi|^2}) & 0 & -i\sqrt{\frac{2}{3}}\xi \\ 0 & -i\sqrt{\frac{2}{3}}\xi^T & 0 \end{pmatrix}, \quad (2.23)$$

which admits five eigenvalues  $\alpha_j(\xi)$  satisfying

$$\alpha_j(\xi) = 0, \quad j = 0, 2, 3, \quad \alpha_{\pm 1}(\xi) = \pm i\sqrt{1 + \frac{5}{3}|\xi|^2}. \quad (2.24)$$

**Lemma 2.5.** *Let  $\xi \neq 0$ , we have for  $A(\xi)$  and  $Q(\xi)$  defined by (2.22) that*

1. *If  $\lambda \neq \alpha_j(\xi)$ , then the operator  $\lambda \mathbf{P}_0 - A(\xi)$  is invertible on  $N_0$  and satisfies*

$$\|(\lambda \mathbf{P}_0 - A(\xi))^{-1}\|_\xi = \max_{-1 \leq j \leq 3} (|\lambda - \alpha_j(\xi)|^{-1}), \quad (2.25)$$

$$\|\mathbf{P}_1(v \cdot \xi)\mathbf{P}_0(\lambda \mathbf{P}_0 - A(\xi))^{-1}\mathbf{P}_0\|_\xi \leq C|\xi| \max_{-1 \leq j \leq 3} (|\lambda - \alpha_j(\xi)|^{-1}), \quad (2.26)$$

where  $\alpha_j(\xi)$ ,  $j = -1, 0, 1, 2, 3$ , are the eigenvalues of  $A(\xi)$  defined by (2.24).

2. *If  $\operatorname{Re} \lambda > -\mu$ , then the operator  $\lambda \mathbf{P}_1 - Q(\xi)$  is invertible on  $N_0^\perp$  and satisfies*

$$\|(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\| \leq (\operatorname{Re} \lambda + \mu)^{-1}, \quad (2.27)$$

$$\|\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1\|_\xi \leq C(1 + |\lambda|)^{-1}[(\operatorname{Re} \lambda + \mu)^{-1} + 1](|\xi| + |\xi|^2). \quad (2.28)$$

*Proof.* Since  $\alpha_j(\xi)$  for  $-1 \leq j \leq 3$  are the eigenvalues of  $A(\xi)$ , it follows that  $\lambda \mathbf{P}_0 - A(\xi)$  is invertible on  $N_0$  for  $\lambda \neq \alpha_j(\xi)$ . By (2.5) we have for  $f, g \in N_0$  that

$$\begin{aligned} (iA(\xi)f, g)_\xi &= ((v \cdot \xi)(f + \frac{1}{|\xi|^2}P_d f), g + \frac{1}{|\xi|^2}P_d g) \\ &= (f + \frac{1}{|\xi|^2}P_d f, (v \cdot \xi)(g + \frac{1}{|\xi|^2}P_d g)) = (f, iA(\xi)g)_\xi. \end{aligned} \quad (2.29)$$

This means that the operator  $iA(\xi)$  is self-adjoint with respect to the inner product  $(\cdot, \cdot)_\xi$  so that

$$\|(\lambda \mathbf{P}_0 - A(\xi))^{-1}\|_\xi = \max_{-1 \leq j \leq 3} (|\lambda - \alpha_j(\xi)|^{-1}).$$

This and the fact that  $\|\mathbf{P}_1(v \cdot \xi)\mathbf{P}_0\| \leq C|\xi|$  imply that

$$\|\mathbf{P}_1(v \cdot \xi)\mathbf{P}_0(\lambda \mathbf{P}_0 - A(\xi))^{-1}\mathbf{P}_0 f\| \leq C|\xi| \max_{-1 \leq j \leq 3} (|\lambda - \alpha_j(\xi)|^{-1}) \|f\|_\xi.$$

Then, we show that for any  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > -\mu$ , the operator  $\lambda \mathbf{P}_1 - Q(\xi) = \lambda \mathbf{P}_1 - L + i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1$  is invertible from  $N_0^\perp$  to itself. Indeed, by (1.12), we obtain for any  $f \in N_0^\perp \cap D(L)$  that

$$\operatorname{Re}[(\lambda \mathbf{P}_1 - L + i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1)f, f] = \operatorname{Re} \lambda (f, f) - (Lf, f) \geq (\mu + \operatorname{Re} \lambda) \|f\|^2, \quad (2.30)$$

which implies that the operator  $\lambda \mathbf{P}_1 - Q(\xi)$  is an injective map from  $N_0^\perp$  to itself so long as  $\operatorname{Re} \lambda > -\mu$ , and its range  $\operatorname{Ran}[\lambda \mathbf{P}_1 - Q(\xi)]$  is a closed subspace of  $L^2(\mathbb{R}_v^3)$ . It then remains to show that the operator  $\lambda \mathbf{P}_1 - Q(\xi)$  is also a surjective map from  $N_0^\perp$  to  $N_0^\perp$ , namely,  $\operatorname{Ran}[\lambda \mathbf{P}_1 - Q(\xi)] = N_0^\perp$ . In fact, if it does not hold, then there exists a function  $g \in N_0^\perp \setminus \operatorname{Ran}[\lambda \mathbf{P}_1 - Q(\xi)]$  with  $g \neq 0$  so that for any  $f \in N_0^\perp \cap D(L)$  that

$$([\lambda \mathbf{P}_1 - L + i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1]f, g) = (f, [\bar{\lambda}\mathbf{P}_1 - L - i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1]g) = 0,$$

which yields  $g = 0$  since the operator  $\bar{\lambda}\mathbf{P}_1 - L - i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1$  is dissipative and satisfies the same estimate as (2.30). This is a contradiction, and thus  $\operatorname{Ran}[\lambda \mathbf{P}_1 - Q(\xi)] = N_0^\perp$ . The estimate (2.27) follows directly from (2.30).

Since it holds

$$(\mathbf{P}_0(v \cdot \omega)\mathbf{P}_1 f, \sqrt{M}) = (\mathbf{P}_1 f, (v \cdot \omega)\sqrt{M}) = 0, \quad \forall f \in L^2(\mathbb{R}_v^3),$$

it follows that  $P_d(\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1) = 0$ . This together with (2.27) and the fact  $\|\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1\| \leq C|\xi|$  lead to

$$\|\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1 f\|_\xi = \|\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1 f\| \leq C(\operatorname{Re} \lambda + \mu)^{-1}|\xi|\|f\|. \quad (2.31)$$

Meanwhile, we can decompose the operator  $\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1$  as

$$\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1 = \frac{1}{\lambda}\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1 + \frac{1}{\lambda}\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1 Q(\xi)(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1.$$

This together with (2.27) and the fact  $\|\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1 Q(\xi)\| \leq C(|\xi| + |\xi|^2)$  give

$$\|\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1 f\|_\xi \leq C|\lambda|^{-1}[(\operatorname{Re} \lambda + \mu)^{-1} + 1](|\xi| + |\xi|^2)\|f\|. \quad (2.32)$$

The combination of the two cases (2.31) and (2.32) yields (2.28).  $\square$

By Lemmas 2.2–2.5, we are able to analyze the spectral and resolvent sets of the operator  $\hat{B}(\xi)$  as follows.

**Lemma 2.6.** *For any constants  $\delta_1 > 0$  and  $\delta_2 > 0$ , there exist two constant  $y_1 = y_1(\delta_1) > 0$  and  $r_2 = r_2(\delta_1, \delta_2) > 0$  so that*

1. *It holds for all  $\xi \neq 0$  that the resolvent set of  $\hat{B}(\xi)$  contains the following domain*

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\mu + \delta_1, |\operatorname{Im} \lambda| \geq y_1\} \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\} \subset \rho(\hat{B}(\xi)). \quad (2.33)$$

2. *It holds for  $0 < |\xi| \leq r_2$  that the spectrum set of  $\hat{B}(\xi)$  is located in the following domain*

$$\sigma(\hat{B}(\xi)) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\mu + \delta_1\} \subset \bigcup_{j=-1}^1 \{\lambda \in \mathbb{C} \mid |\lambda - \alpha_j(\xi)| \leq \delta_2\}, \quad (2.34)$$

where  $\alpha_j(\xi)$ ,  $j = -1, 0, 1$ , are the eigenvalues of  $A(\xi)$  defined in (2.24).

*Proof.* By Lemmas 2.5, we have for  $\operatorname{Re} \lambda > -\mu$  and  $\lambda \neq \alpha_j(\xi)$  ( $-1 \leq j \leq 3$ ) that the operator  $\lambda \mathbf{P}_0 - A(\xi) + \lambda \mathbf{P}_1 - Q(\xi)$  is invertible on  $L_\xi^2(\mathbb{R}_v^3)$  and it satisfies

$$(\lambda \mathbf{P}_0 - A(\xi) + \lambda \mathbf{P}_1 - Q(\xi))^{-1} = (\lambda \mathbf{P}_0 - A(\xi))^{-1}\mathbf{P}_0 + (\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1,$$

because the operator  $\lambda \mathbf{P}_0 - A(\xi)$  is orthogonal to  $\lambda \mathbf{P}_1 - Q(\xi)$ . Therefore, we can re-write (2.21) as

$$\begin{aligned} \lambda - \hat{B}(\xi) &= Y_0(\lambda, \xi)((\lambda \mathbf{P}_0 - A(\xi)) + (\lambda \mathbf{P}_1 - Q(\xi))), \\ Y_0(\lambda, \xi) &=: I + i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_0(\lambda \mathbf{P}_0 - A(\xi))^{-1}\mathbf{P}_0 + i\mathbf{P}_0(v \cdot \xi)\mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1}\mathbf{P}_1. \end{aligned}$$

As shown in the proof of Lemma 2.4, there exists  $r_1 = r_1(\delta_1) > 0$  so that  $\rho(\hat{B}(\xi)) \supset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\nu_0 + \delta_1\}$  for  $|\xi| > r_1$ . For the case  $|\xi| \leq r_1$ , by (2.26) and (2.28) we can choose  $y_1 = y_1(\delta_1) > 0$  such that it holds for  $\operatorname{Re} \lambda \geq -\mu + \delta_1$  and  $|\operatorname{Im} \lambda| \geq y_1$  that

$$\|\mathbf{P}_1(v \cdot \xi) \mathbf{P}_0(\lambda \mathbf{P}_0 - A(\xi))^{-1} \mathbf{P}_0\|_\xi \leq \frac{1}{4}, \quad \|\mathbf{P}_0(v \cdot \xi) \mathbf{P}_1(\lambda \mathbf{P}_1 - Q(\xi))^{-1} \mathbf{P}_1\|_\xi \leq \frac{1}{4}. \quad (2.35)$$

This implies that the operator  $Y_0(\lambda, \xi)$  is invertible on  $L_\xi^2(\mathbb{R}_v^3)$  and thus  $\lambda - \hat{B}(\xi)$  is invertible on  $L_\xi^2(\mathbb{R}_v^3)$  and satisfies

$$(\lambda - \hat{B}(\xi))^{-1} = [(\lambda \mathbf{P}_0 - A(\xi))^{-1} \mathbf{P}_0 + (\lambda \mathbf{P}_1 - Q(\xi))^{-1} \mathbf{P}_1] Y_0(\lambda, \xi)^{-1}. \quad (2.36)$$

Therefore,  $\rho(\hat{B}(\xi)) \supset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\mu + \delta_1, |\operatorname{Im} \lambda| \geq y_1\}$  for  $|\xi| \leq r_1$ . This and Lemma 2.2 lead to (2.33).

Assume that  $\min_{-1 \leq j \leq 1} |\lambda - \alpha_j(\xi)| > \delta_2$  and  $\operatorname{Re} \lambda \geq -\mu + \delta_1$ . Then, by (2.26) and (2.28) we can choose  $r_2 = r_2(\delta_1, \delta_2) > 0$  so that estimates (2.35) still hold for  $0 < |\xi| \leq r_2$ , and the operator  $\lambda - \hat{B}(\xi)$  is invertible on  $L_\xi^2(\mathbb{R}^3)$ . Therefore, we have  $\rho(\hat{B}(\xi)) \supset \{\lambda \in \mathbb{C} \mid \min_{-1 \leq j \leq 1} |\lambda - \alpha_j(\xi)| > \delta_2, \operatorname{Re} \lambda \geq -\mu + \delta_1\}$  for  $0 < |\xi| \leq r_2$ , which gives (2.34).  $\square$

## 2.2 Low frequency asymptotics of eigenvalues

We study the low frequency asymptotics of the eigenvalues and eigenfunctions of the operator  $\hat{B}(\xi)$  in this subsection. In terms of (2.3), the eigenvalue problem  $\hat{B}(\xi)f = \lambda f$  can be written as

$$\lambda f = (L - i(v \cdot \xi))f - \frac{i(v \cdot \xi)\sqrt{M}}{|\xi|^2} \int_{\mathbb{R}^3} f \sqrt{M} dv, \quad |\xi| \neq 0. \quad (2.37)$$

By macro-micro decomposition, the eigenfunction  $f$  of (2.37) can be divided into

$$f = f_0 + f_1 =: \mathbf{P}_0 f + \mathbf{P}_1 f = \mathbf{P}_0 f + (I - \mathbf{P}_0)f.$$

Hence (2.37) gives

$$\lambda f_0 = -\mathbf{P}_0[i(v \cdot \xi)(f_0 + f_1)] - \frac{i(v \cdot \xi)\sqrt{M}}{|\xi|^2} \int_{\mathbb{R}^3} f_0 \sqrt{M} dv, \quad (2.38)$$

$$\lambda f_1 = L f_1 - \mathbf{P}_1[i(v \cdot \xi)(f_0 + f_1)] \Leftrightarrow (\lambda \mathbf{P}_1 - Q(\xi))f_1 = -i\mathbf{P}_1(v \cdot \xi)f_0. \quad (2.39)$$

By Lemma 2.5, (2.22) and (2.39), the microscopic part  $f_1$  can be represented in terms of the macroscopic part  $f_0$  for all  $\operatorname{Re} \lambda > -\mu$  as

$$f_1 = -i(\lambda \mathbf{P}_1 - Q(\xi))^{-1} \mathbf{P}_1(v \cdot \xi)f_0, \quad \operatorname{Re} \lambda > -\mu. \quad (2.40)$$

Substituting it into (2.38), we obtain the eigenvalue problem for macroscopic part  $f_0$  as

$$\lambda f_0 = -i\mathbf{P}_0(v \cdot \xi)f_0 - \frac{i(v \cdot \xi)\sqrt{M}}{|\xi|^2} \int_{\mathbb{R}^3} f_0 \sqrt{M} dv + \mathbf{P}_0[(v \cdot \xi)R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)f_0], \quad \operatorname{Re} \lambda > -\mu, \quad (2.41)$$

where  $R(\lambda, \xi)$  is the resolvent of the operator  $Q(\xi)$  for  $\operatorname{Re} \lambda > -\mu$  defined by

$$R(\lambda, \xi) = -(\lambda \mathbf{P}_1 - Q(\xi))^{-1} = [L - \lambda \mathbf{P}_1 - i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1]^{-1}. \quad (2.42)$$

To solve the eigenvalue problem (2.41), we write  $f_0 \in N_0$  in terms of the basis  $\chi_j$  as

$$f_0 = \sum_{j=0}^4 W_j \chi_j \quad \text{with} \quad W_j = (f, \chi_j), \quad j = 0, 1, 2, 3, 4, \quad (2.43)$$

with the unknown coefficients  $(W_0, W_1, W_2, W_3, W_4)$  to be determined below. Taking the inner product between (2.41) and  $\chi_j$  for  $j = 0, 1, 2, 3, 4$  respectively, we have the equations about  $\lambda$  and  $(W_0, W, W_4)$  with  $W = (W_1, W_2, W_3)$  for  $\text{Re}\lambda > -\mu$ :

$$\lambda W_0 = -i(W \cdot \xi) =: -i \sum_{i=1}^3 W_i \xi_i, \quad (2.44)$$

$$\begin{aligned} \lambda W_i = & -iW_0 \left( \xi_i + \frac{\xi_i}{|\xi|^2} \right) - i\sqrt{\frac{2}{3}}W_4\xi_i + \sum_{j=1}^3 W_j(R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_j, (v \cdot \xi)\chi_i) \\ & + W_4(R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_4, (v \cdot \xi)\chi_i), \end{aligned} \quad (2.45)$$

$$\begin{aligned} \lambda W_4 = & -i\sqrt{\frac{2}{3}}(W \cdot \xi) + \sum_{j=1}^3 W_j(R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_j, (v \cdot \xi)\chi_4) \\ & + W_4(R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_4, (v \cdot \xi)\chi_4). \end{aligned} \quad (2.46)$$

We apply the following transform so as to simplify the system (2.44)-(2.46).

**Lemma 2.7.** *Let  $e_1 = (1, 0, 0)$ ,  $\xi = s\omega$  with  $s \in \mathbb{R}$ ,  $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$ . Then, it holds for  $1 \leq i, j \leq 3$  and  $\text{Re}\lambda > -\mu$  that*

$$\begin{aligned} (R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_j, (v \cdot \xi)\chi_i) = & s^2(\delta_{ij} - \omega_i\omega_j)(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_2), v_1\chi_2) \\ & + s^2\omega_i\omega_j(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_1), \end{aligned} \quad (2.47)$$

$$(R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_4, (v \cdot \xi)\chi_i) = s^2\omega_i(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_1), \quad (2.48)$$

$$(R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_i, (v \cdot \xi)\chi_4) = s^2\omega_i(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_4), \quad (2.49)$$

$$(R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_4, (v \cdot \xi)\chi_4) = s^2(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_4). \quad (2.50)$$

*Proof.* Let  $\mathbb{O}$  be an orthogonal transformation of  $\mathbb{R}^3$ , and denote  $(\mathbb{O}f)(v) = f(\mathbb{O}v)$ . Recalling the definition (1.8)

$$(Lf)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \omega) (\sqrt{M'_*}f' + \sqrt{M'}f'_* - \sqrt{M}f_* - \sqrt{M_*}f) \sqrt{M_*} d\omega dv_*,$$

and making the variable transforms  $v \rightarrow \mathbb{O}v$ ,  $v_* \rightarrow \mathbb{O}v_*$  and  $\omega \rightarrow \mathbb{O}\omega$  which imply  $v' \rightarrow \mathbb{O}v'$  and  $v'_* \rightarrow \mathbb{O}v'_*$ , we can prove  $(Lf)(\mathbb{O}v) = L(\mathbb{O}f)(v)$ ,  $\mathbf{P}_0 f(\mathbb{O}v) = \mathbf{P}_0(\mathbb{O}f)(v)$ , and  $[R(\lambda, \xi)f](\mathbb{O}v) = R(\lambda, \mathbb{O}^T\xi)(\mathbb{O}f)(v)$  by straightforward computation.

For any given  $\xi \neq 0$ , we choose  $\mathbb{O}$  to be a rotation transform of  $\mathbb{R}^3$  satisfying  $\mathbb{O}^T\xi = se_1$ , from which we have  $\mathbb{O}_{i1} = \omega_i$ . Take the variable transform  $v = \mathbb{O}u$  so that  $v \cdot \xi = u \cdot \mathbb{O}^T\xi = su_1$ , we have

$$\begin{aligned} (R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)v_j\sqrt{M}, (v \cdot \xi)v_i\sqrt{M}) = & (R(\lambda, se_1)\mathbf{P}_1(su_1(\sum_{k=1}^3 \mathbb{O}_{jk}u_k)\sqrt{M}), su_1(\sum_{l=1}^3 \mathbb{O}_{il}u_l)\sqrt{M})) \\ = & s^2 \sum_{k,l=1}^3 \mathbb{O}_{jk}\mathbb{O}_{il}(R(\lambda, se_1)\mathbf{P}_1(u_1u_k\sqrt{M}, u_1u_l\sqrt{M})). \end{aligned} \quad (2.51)$$

To deal with the right hand side of (2.51), we assume without the loss of generality that  $l \neq 1$  if  $k \neq l$ . By changing variable  $w_l = -u_l$ ,  $w_j = u_j$  ( $j \neq l$ ), we have

$$(R(\lambda, se_1)\mathbf{P}_1(u_1u_k\sqrt{M}, u_1u_l\sqrt{M}) = -(R(\lambda, se_1)\mathbf{P}_1(w_1w_k\sqrt{M}, w_1w_l\sqrt{M}),$$

where we have used the fact that  $R(\lambda, se_1)$  is invariant under any rotation transform  $\mathbb{O}$  with  $\mathbb{O}e_1 = e_1$ . This implies

$$(R(\lambda, se_1)\mathbf{P}_1(u_1u_k\sqrt{M}, u_1u_l\sqrt{M}) = 0, \quad \text{for } k \neq l.$$

If  $k = l = 3$ , by changing variable  $w_2 = u_3$ ,  $w_3 = u_2$ ,  $w_1 = u_1$ , we have

$$(R(\lambda, se_1)\mathbf{P}_1(u_1u_3\sqrt{M}), u_1u_3\sqrt{M}) = (R(\lambda, se_1)\mathbf{P}_1(w_1w_2\sqrt{M}), w_1w_2\sqrt{M}).$$

The combination of the above two cases yields (2.47).

Applying the above variable transform  $v = \mathbb{O}u$  again, we have

$$\begin{aligned} (R(\lambda, \xi)\mathbf{P}_1(v \cdot \xi)\chi_4, (v \cdot \xi)v_i\sqrt{M}) &= (R(\lambda, se_1)\mathbf{P}_1(su_1)\chi_4, su_1(\sum_{k=1}^3 \mathbb{O}_{ik}u_k)\sqrt{M}) \\ &= s^2 \sum_{k=1}^3 \mathbb{O}_{ik} (R(\lambda, se_1)\mathbf{P}_1(u_1\chi_4), u_1u_k\sqrt{M}), \end{aligned}$$

which leads to (2.48) since it can be shown that  $(R(\lambda, se_1)\mathbf{P}_1(u_1\chi_4), u_1u_k\sqrt{M}) = 0$  if  $k \neq 1$  after changing variable  $w_k = -u_k$  and  $w_j = u_j$  with  $j \neq k$ . Similar argument yields (2.49) and (2.50).  $\square$

With the help of (2.47)-(2.50), the equations (2.44)-(2.46) can be simplified as

$$\lambda W_0 = -is(W \cdot \omega), \quad (2.52)$$

$$\begin{aligned} \lambda W_i &= -iW_0 \left( s + \frac{1}{s} \right) \omega_i - is\sqrt{\frac{2}{3}}W_4\omega_i + s^2(W \cdot \omega)\omega_i(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_1) \\ &\quad + s^2(W_i - (W \cdot \omega)\omega_i)(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_2), v_1\chi_2) \\ &\quad + s^2W_4\omega_i(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_1), \quad i = 1, 2, 3, \end{aligned} \quad (2.53)$$

$$\begin{aligned} \lambda W_4 &= -is\sqrt{\frac{2}{3}}(W \cdot \omega) + s^2(W \cdot \omega)(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_4) \\ &\quad + s^2W_4(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_4). \end{aligned} \quad (2.54)$$

Multiplying (2.53) by  $\omega_i$  and making the summation of resulted equations, we have

$$\begin{aligned} \lambda(W \cdot \omega) &= -iW_0 \left( s + \frac{1}{s} \right) - is\sqrt{\frac{2}{3}}W_4 + s^2(W \cdot \omega)(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_1) \\ &\quad + s^2W_4(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_1). \end{aligned} \quad (2.55)$$

Furthermore, we multiply (2.55) by  $\omega_i$  and subtract the resulted equation from (2.53) to have

$$(W_i - (W \cdot \omega)\omega_i)(\lambda - s^2(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_2), v_1\chi_2)) = 0, \quad i = 1, 2, 3. \quad (2.56)$$

Denote by  $U = (W_0, W \cdot \omega, W_4)$  a vector in  $\mathbb{R}^3$ . The system (2.52), (2.54) and (2.55) can be written as  $\mathbb{M}U = 0$  with the matrix  $\mathbb{M}$  defined by

$$\mathbb{M} = \begin{pmatrix} \lambda & is & 0 \\ i(s + \frac{1}{s}) & \lambda - s^2a_{11} & is\sqrt{\frac{2}{3}} - s^2a_{41} \\ 0 & is\sqrt{\frac{2}{3}} - s^2a_{14} & \lambda - s^2a_{44} \end{pmatrix}, \quad (2.57)$$

with  $a_{ij} = (R(\lambda, se_1)\mathbf{P}_1(v_1\chi_i), v_1\chi_j)$ . The equation  $\mathbb{M}U = 0$  admits a non-trivial solution  $U \neq 0$  for  $\text{Re}\lambda > -\mu$  if and only if it holds  $\det(\mathbb{M}) = 0$  for  $\text{Re}\lambda > -\mu$ . To solve this equation, we need to investigate the eigenvalues of the matrix  $\mathbb{M}$ . Denote

$$D_0(\lambda, s) =: \lambda - s^2(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_2), v_1\chi_2). \quad (2.58)$$

Then, by direct computation and the implicit function theorem, we can show

**Lemma 2.8.** *The equation  $D_0(\lambda, s) = 0$  has a unique  $C^\infty$  solution  $\lambda = \lambda(s)$  for  $(s, \lambda) \in [-r_0, r_0] \times B_{r_1}(0)$  with  $r_0, r_1 > 0$  being small constants that satisfies*

$$\lambda(0) = 0, \quad \lambda'(0) = 0, \quad \lambda''(0) = 2(L^{-1}\mathbf{P}_1(v_1\chi_2), v_1\chi_2).$$

We have the following result about the eigenvalues of the matrix  $\mathbb{M}$ .

**Lemma 2.9.** *There exist two small constants  $r_0 > 0$  and  $r_1 > 0$  so that the equation  $D(\lambda, s) =: \det(\mathbb{M}) = 0$  admits exactly one  $C^\infty$  solution  $\lambda_j(s)$  ( $j = -1, 0, 1$ ) for  $(s, \lambda_j) \in [-r_0, r_0] \times B_{r_1}(ji)$  that satisfy*

$$\lambda_j(0) = ji, \quad \lambda'_j(0) = 0, \quad (2.59)$$

$$\begin{aligned} \lambda''_{\pm 1}(0) = & (L(L + i\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1), (L + i\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1)) \\ & \pm i(\|(L + i\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1)\|^2 + \frac{5}{3}), \end{aligned} \quad (2.60)$$

$$\lambda''_0(0) = 2(L^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4). \quad (2.61)$$

Moreover,  $\lambda_j(s)$  is an even function and satisfies

$$\overline{\lambda_j(s)} = \lambda_{-j}(-s) = \lambda_{-j}(s) \quad \text{for } j = 0, \pm 1. \quad (2.62)$$

In particular,  $\lambda_0(s)$  is a real function.

*Proof.* By direct computation, we have

$$\begin{aligned} D(\lambda, s) = & \lambda^3 - \lambda^2 s^2 [(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_1) + (R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_4)] \\ & + \lambda \left\{ 1 + \frac{5}{3}s^2 + i\sqrt{\frac{2}{3}}s^3 [(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_1) + (R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_4)] \right. \\ & \quad + s^4 (R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_4)(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_1) \\ & \quad \left. - s^4 (R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_1)(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_4) \right\} \\ & - (s^2 + s^4)(R(\lambda, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_4). \end{aligned} \quad (2.63)$$

(2.63) has three roots of the form  $(\lambda, s) = (ji, 0)$  for  $j = -1, 0, 1$ , with  $\lambda = ji$  being the solution to  $D(\lambda, 0) = \lambda(\lambda^2 + 1)$ . Since for each  $ji$ ,  $j = 0, \pm 1$ , it holds

$$\partial_s D(ji, 0) = 0, \quad \partial_\lambda D(ji, 0) = 1 - 3j^2 \neq 0, \quad (2.64)$$

the implicit function theorem implies that there exists small constants  $r_0, r_1 > 0$  and a unique  $C^\infty$  function  $\lambda_j(s): [-r_0, r_0] \rightarrow B_{r_1}(ji)$  so that  $D(\lambda_j(s), s) = 0$  for  $s \in [-r_0, r_0]$ , and in particular

$$\lambda_j(0) = ji \quad \text{and} \quad \lambda'_j(0) = -\frac{\partial_s D(ji, 0)}{\partial_\lambda D(ji, 0)} = 0, \quad j = 0, \pm 1. \quad (2.65)$$

Direct computation gives

$$\begin{aligned} \partial_s^2 D(ji, 0) = & 2j^2 [(L - ji\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1), v_1\chi_1) + ((L - ji\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4)] \\ & + \frac{10}{3}ji - 2((L - ji\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4), \end{aligned}$$

which together with (2.64) yields

$$\begin{cases} \lambda''_0(0) = -\frac{\partial_s^2 D(0, 0)}{\partial_\lambda D(0, 0)} = 2(L^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4), \\ \lambda''_{\pm 1}(0) = -\frac{\partial_s^2 D(\pm i, 0)}{\partial_\lambda D(\pm i, 0)} = ((L \mp i\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1), v_1\chi_1) \pm \frac{5}{3}i. \end{cases} \quad (2.66)$$

Thus, the properties (2.59)–(2.61) follow from (2.65)–(2.66) and  $\overline{\lambda''_1(0)} = \lambda''_{-1}(0)$ .

Finally, since by (2.63), it holds that  $D(\lambda, s) = D(\lambda, -s)$ ,  $\overline{D(\lambda, s)} = D(\overline{\lambda}, -s)$ , we can obtain (2.62) by using the fact that  $\lambda_j(s) = ji + O(s^2)$  as  $s \rightarrow 0$ .  $\square$

**Remark 2.10.** In general, the electric potential equation in (1.5) takes the form  $\varepsilon^2 \Delta_x \Phi = \int_{\mathbb{R}^3} f \sqrt{M} dv$  with  $\varepsilon > 0$ . Then similar to the above lemma, we can prove that there exists a constant  $r_0(\varepsilon) > 0$  such that the equation  $D(\lambda, s) = 0$  has exactly three solutions  $\lambda_j(s)$  ( $j = 0, \pm 1$ ) for  $|s| \leq r_0(\varepsilon)$ , which satisfy  $\lambda_j(0) = \frac{j}{\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow \infty$ .

With the help of Lemmas 2.7–2.9, we are able to construct the eigenfunction  $\psi_j(s, \omega)$  corresponding to the eigenvalue  $\lambda_j$  at the low frequency. Indeed, we have

**Theorem 2.11.** There exists a constant  $r_0 > 0$  so that the spectrum  $\lambda \in \sigma(B(\xi)) \subset \mathbb{C}$  for  $\xi = s\omega$  with  $|s| \leq r_0$  and  $\omega \in \mathbb{S}^2$  consists of five points  $\{\lambda_j(s), j = -1, 0, 1, 2, 3\}$  on the domain  $\text{Re } \lambda > -\mu/2$ . The spectrum  $\lambda_j(s)$  and the corresponding eigenfunction  $\psi_j(s, \omega)$  are  $C^\infty$  functions of  $s$  for  $|s| \leq r_0$ . In particular, the eigenvalues admit the following asymptotic expansion for  $|s| \leq r_0$

$$\begin{cases} \lambda_{\pm 1}(s) = \pm i + (-a_1 \pm ib_1)s^2 + o(s^2), & \overline{\lambda_1} = \lambda_{-1}, \\ \lambda_0(s) = -a_0s^2 + o(s^2), \\ \lambda_2(s) = \lambda_3(s) = -a_2s^2 + o(s^2), \end{cases} \quad (2.67)$$

where  $a_j > 0$  ( $0 \leq j \leq 2$ ) and  $b_1 > 0$  are defined by

$$\begin{cases} a_1 = -\frac{1}{2}(L(L + i\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1), (L + i\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1)), \\ b_1 = \frac{1}{2}(\|(L + i\mathbf{P}_1)^{-1}\mathbf{P}_1(v_1\chi_1)\|^2 + \frac{5}{3}), \\ a_0 = -(L^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4), \quad a_2 = -(L^{-1}\mathbf{P}_1(v_1\chi_2), v_1\chi_2). \end{cases} \quad (2.68)$$

The eigenfunctions are orthogonal to each other and satisfy

$$\begin{cases} (\psi_j(s, \omega), \overline{\psi_k(s, \omega)})_\xi = \delta_{jk}, \quad j, k = -1, 0, 1, 2, 3, \\ \psi_j(s, \omega) = \psi_{j,0}(\omega) + \psi_{j,1}(\omega)s + \psi_{j,2}(\omega)s^2 + o(s^2), \quad |s| \leq r_0, \end{cases} \quad (2.69)$$

where the coefficients  $\psi_{j,n}$  are given by

$$\begin{cases} \psi_{0,0} = \chi_4, \quad \psi_{0,1} = iL^{-1}\mathbf{P}_1(v \cdot \omega)\chi_4, \quad (\psi_{0,2}, \sqrt{M}) = -\sqrt{\frac{2}{3}}, \\ \psi_{\pm 1,0} = \frac{\sqrt{2}}{2}(v \cdot \omega)\sqrt{M}, \quad (\psi_{\pm 1,2}, \sqrt{M}) = 0, \\ \psi_{\pm 1,1} = \mp \frac{\sqrt{2}}{2}\sqrt{M} \mp \frac{\sqrt{3}}{3}\chi_4 + \frac{\sqrt{2}}{2}i(L \mp i\mathbf{P}_1)^{-1}\mathbf{P}_1(v \cdot \omega)^2\sqrt{M}, \\ \psi_{j,0} = (v \cdot c^j)\sqrt{M}, \quad (\psi_{j,n}, \sqrt{M}) = (\psi_{j,n}, \chi_4) = 0 \quad (n \geq 0), \\ \psi_{j,1} = iL^{-1}\mathbf{P}_1[(v \cdot \omega)(v \cdot c^j)\sqrt{M}], \quad j = 2, 3. \end{cases} \quad (2.70)$$

Here,  $c^j$  ( $j = 2, 3$ ) are orthonormal vectors satisfying  $c^j \cdot \omega = 0$ .

**Remark 2.12.** Different from the asymptotical behaviors of the eigenvalues of the linearized Vlasov-Poisson-Boltzmann operator shown in Theorem 2.11, the eigenvalues of the linearized Boltzmann operator  $\hat{E}(\xi) := L - i(v \cdot \xi)$  have the following Taylor expansions for  $|\xi| \leq r_1$  with  $r_1 > 0$  a constant (cf. [7])

$$\begin{cases} \lambda_{\pm 1}(s) = \pm i\sqrt{\frac{5}{3}}s - a_{\pm 1}s^2 + o(s^2), \\ \lambda_0(s) = -a_0s^2 + o(s^2), \\ \lambda_j(s) = -a_js^2 + o(s^2), \quad j = 2, 3, \end{cases} \quad (2.71)$$

where  $a_j > 0$  ( $-1 \leq j \leq 3$ ) are defined by

$$\begin{cases} a_{\pm 1} = -\frac{1}{5}(L^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4) - \frac{1}{2}(L^{-1}\mathbf{P}_1(v_1\chi_1), v_1\chi_1), \\ a_0 = -\frac{3}{5}(L^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4), \quad a_2 = a_3 = -(L^{-1}\mathbf{P}_1(v_1\chi_2), v_1\chi_2). \end{cases} \quad (2.72)$$

**Remark 2.13.** The operator  $A(\xi)$  defined by (2.22) corresponds exactly to that of the linearized Euler-Poisson system after taking Fourier transform. Let  $\psi_j(\xi)$ ,  $-1 \leq j \leq 3$ , be the eigenfunction of  $A(\xi)$ . Then we can obtain for  $\xi = s\omega$

$$\begin{aligned}\psi_0(\xi) &= \left( \frac{\sqrt{\frac{2}{3}}s^2}{\sqrt{1 + \frac{5}{3}s^2}\sqrt{1 + s^2}}, 0, 0, 0, \frac{\sqrt{s^2 + 1}}{\sqrt{1 + \frac{5}{3}s^2}} \right), \\ \psi_{\pm 1}(\xi) &= \frac{\sqrt{2}}{2} \left( \frac{s}{\sqrt{1 + \frac{5}{3}s^2}}, \mp\omega_1, \mp\omega_2, \mp\omega_3, \frac{\sqrt{\frac{2}{3}}s}{\sqrt{1 + \frac{5}{3}s^2}} \right), \\ \psi_j(\xi) &= (0, W_1^j, W_2^j, W_3^j, 0), \quad j = 2, 3,\end{aligned}$$

where  $W^j = (W_1^j, W_2^j, W_3^j)$  satisfies  $W^j \cdot \omega = W^i \cdot W^j = 0$  and  $|W^j| = 1$  for  $2 \leq i \neq j \leq 3$ . It can be seen that  $\psi_j(\xi)$  is not orthonormal to each other with the inner product  $(\cdot, \cdot)$ , but they obey the orthonormal relation as

$$(\psi_i(\xi), \psi_j(\xi))_\xi = \delta_{ij}, \quad -1 \leq i, j \leq 3.$$

*Proof.* The eigenvalues  $\lambda_j(s)$  and the eigenfunctions  $\psi_j(s, \omega)$  can be constructed as follows. For  $j = 2, 3$ , we take  $\lambda_j = \lambda(s)$  to be the solution of the equation  $D_0(\lambda, s) = 0$  defined in Lemma 2.8, and choose  $W_0 = 0$ ,  $W_4 = 0$ , and  $W^j$  to be the linearly independent vector so that  $W^j \cdot \omega = 0$  and  $W^2 \cdot W^3 = 0$ . And the corresponding eigenfunctions  $\psi_2(s, \omega)$  and  $\psi_3(s, \omega)$  are defined by

$$\psi_j(s, \omega) = (W^j \cdot v)\sqrt{M} + \text{is}[L - \lambda_j \mathbf{P}_1 - \text{is}\mathbf{P}_1(v \cdot \omega)\mathbf{P}_1]^{-1}\mathbf{P}_1[(v \cdot \omega)(W^j \cdot v)\sqrt{M}], \quad (2.73)$$

which are orthonormal, i.e.,  $(\psi_2(s, \omega), \overline{\psi_3(s, \omega)})_\xi = 0$ .

For  $j = -1, 0, 1$ , we choose  $\lambda_j = \lambda_j(s)$  to be a solution of  $D(\lambda, s) = 0$  given by Lemma 2.9, and denote by  $\{a_j, b_j, d_j\} =: \{W_0^j, (W \cdot \omega)^j, W_4^j\}$  a solution of system (2.52), (2.54), and (2.55) for  $\lambda = \lambda_j(s)$ . Then we can construct  $\psi_j(s, \omega)$  ( $j = -1, 0, 1$ ) as

$$\begin{cases} \psi_j(s, \omega) = \mathbf{P}_0\psi_j(s, \omega) + \mathbf{P}_1\psi_j(s, \omega), \\ \mathbf{P}_0\psi_j(s, \omega) = a_j(s)\chi_0 + b_j(s)(v \cdot \omega)\sqrt{M} + d_j(s)\chi_4, \\ \mathbf{P}_1\psi_j(s, \omega) = \text{is}[L - \lambda_j \mathbf{P}_1 - \text{is}\mathbf{P}_1(v \cdot \omega)\mathbf{P}_1]^{-1}\mathbf{P}_1[(v \cdot \omega)\mathbf{P}_0\psi_j(s, \omega)]. \end{cases} \quad (2.74)$$

We write

$$(L - \text{is}(v \cdot \omega) - \frac{\text{i}}{s}(v \cdot \omega)P_d)\psi_j(s, \omega) = \lambda_j(s)\psi_j(s, \omega), \quad -1 \leq j \leq 3.$$

Taking the inner product  $(\cdot, \cdot)_\xi$  of the above equation with  $\overline{\psi_j(s, \omega)}$  and using the facts that

$$\begin{aligned}(\hat{B}(\xi)f, g)_\xi &= (f, \hat{B}(-\xi)g)_\xi, \quad f, g \in L_\xi^2(\mathbb{R}^3) \cap D(\hat{B}(\xi)), \\ (L + \text{is}(v \cdot \omega) + \frac{\text{i}}{s}(v \cdot \omega)P_d)\overline{\psi_j(s, \omega)} &= \overline{\lambda_j(s)} \cdot \overline{\psi_j(s, \omega)},\end{aligned}$$

we have

$$(\lambda_j(s) - \overline{\lambda_k(s)})(\psi_j(s, \omega), \overline{\psi_k(s, \omega)})_\xi = 0, \quad -1 \leq j, k \leq 3.$$

For  $s \neq 0$  being sufficiently small,  $\lambda_j(s) \neq \overline{\lambda_k(s)}$  for  $-1 \leq j \neq k \leq 1$  and  $j = 0, \pm 1$ ,  $k = 2, 3$ . Therefore, we have

$$(\psi_j(s, \omega), \overline{\psi_k(s, \omega)})_\xi = 0, \quad -1 \leq j \neq k \leq 3.$$

We can normalize them by taking  $(\psi_j(s, \omega), \overline{\psi_j(s, \omega)})_\xi = 1$  for  $-1 \leq j \leq 3$ .

To investigate the asymptotic expression of eigenfunctions at the low frequency, we take the Taylor expansion for both eigenvalues and eigenfunctions as

$$\lambda_j(s) = \sum_{n=0}^2 \lambda_{j,n} s^n + O(s^3), \quad \psi_j(s, \omega) = \sum_{n=0}^2 \psi_{j,n}(\omega) s^n + O(s^3)$$



Substituting the above expansion into (2.73), we obtain the expansion of  $\psi_j(s, \omega)$  for  $j = 2, 3$  in (2.70).

To obtain expansion of  $\psi_j(s, \omega)$  for  $j = 0, \pm 1$  defined in (2.74), we deal with its macroscopic part and microscopic part respectively. The expansion of macroscopic part  $\mathbf{P}_0(\psi_j(s, \omega))$  is determined in terms of the coefficients  $\{a_j(s), b_j(s), d_j(s)\}$  that satisfy

$$\left\{ \begin{array}{l} \lambda_j(s)a_j(s) + isb_j(s) = 0, \\ i(s^2 + 1)a_j(s) + (s\lambda_j(s) - s^3(R(\lambda_j, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_1))b_j(s) \\ \quad + (is^2\sqrt{\frac{2}{3}} - s^3(R(\lambda_j, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_1))d_j(s) = 0, \\ (is\sqrt{\frac{2}{3}} - s^2(R(\lambda_j, se_1)\mathbf{P}_1(v_1\chi_1), v_1\chi_4))b_j(s) \\ \quad + (\lambda_j(s) - s^2(R(\lambda_j, se_1)\mathbf{P}_1(v_1\chi_4), v_1\chi_4))d_j(s) = 0. \end{array} \right. \quad (2.75)$$

Assume

$$a_j(s) = \sum_{n=0}^2 a_{j,n}s^n + O(s^3), \quad b_j(s) = \sum_{n=0}^2 b_{j,n}s^n + O(s^3), \quad d_j(s) = \sum_{n=0}^2 d_{j,n}s^n + O(s^3).$$

Substituting the above expansion into (2.75) and (2.74), we can obtain the expansion of  $\psi_j(s, \omega)$  for  $j = -1, 0, 1$  given in (2.70) after a tedious but straightforward computation. Hence, we omit the detail for brevity.  $\square$

### 3 Optimal time-decay rates of linearized VPB

In this section, we consider the Cauchy problem (2.1) for the linearized Vlasov-Poisson-Boltzmann equations and establish the optimal time-decay rates of global solution based on the results obtained in Sect. 2.

#### 3.1 Decomposition and asymptotics of linear semigroup

We start by proving

**Lemma 3.1.** *The operator  $Q(\xi) = L - i\mathbf{P}_1(v \cdot \xi)\mathbf{P}_1$  generates a strongly continuous contraction semigroup on  $N_0^\perp$  for any fixed  $|\xi| \neq 0$ , which satisfies for any  $t > 0$  and  $f \in N_0^\perp \cap L^2(\mathbb{R}_v^3)$  that*

$$\|e^{tQ(\xi)}f\| \leq e^{-\mu t}\|f\|. \quad (3.1)$$

In addition, for any  $x > -\mu$  and  $f \in N_0^\perp \cap L^2(\mathbb{R}_v^3)$  it holds

$$\int_{-\infty}^{+\infty} \|[(x + iy)\mathbf{P}_1 - Q(\xi)]^{-1}f\|^2 dy \leq \pi(x + \mu)^{-1}\|f\|^2. \quad (3.2)$$

*Proof.* Since the operator  $Q(\xi)$  is a densely defined closed operator on  $N_0^\perp$ , and both  $Q(\xi)$  and  $Q(\xi)^* = Q(-\xi)$  are dissipative on  $N_0^\perp$  and satisfy (2.27), it follows from Lemma 6.2 that  $Q(\xi)$  generates a strongly continuous contraction semigroup on  $N_0^\perp$  and satisfies (3.1).

The resolvent  $(\lambda - Q(\xi))^{-1}$  can be expressed for  $\lambda \in \rho(Q(\xi))$  as

$$[\lambda\mathbf{P}_1 - Q(\xi)]^{-1} = \int_0^\infty e^{-\lambda t} e^{tQ(\xi)} dt, \quad \operatorname{Re} \lambda > -\mu,$$

and for  $\lambda = x + iy$  as

$$[(x + iy)\mathbf{P}_1 - Q(\xi)]^{-1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iyt} \left[ \sqrt{2\pi} 1_{\{t \geq 0\}} e^{-xt} e^{tQ(\xi)} \right] dt,$$

where the right hand side is the Fourier transform of the function  $\sqrt{2\pi}1_{\{t \geq 0\}}e^{-xt}e^{tQ(\xi)}$  with respect to the time variable. By Parseval's equality, we have for  $f \in N_0^\perp$  that

$$\begin{aligned} \int_{-\infty}^{+\infty} \|(x+iy)\mathbf{P}_1 - Q(\xi)]^{-1}f\|^2 dy &= \int_{-\infty}^{+\infty} \|(2\pi)^{\frac{1}{2}}1_{\{t \geq 0\}}e^{-xt}e^{tQ(\xi)}f\|^2 dt \\ &= 2\pi \int_0^\infty e^{-2xt} \|e^{tQ(\xi)}f\|^2 dt \leq 2\pi \int_0^\infty e^{-2(x+\mu)t} dt \|f\|^2, \end{aligned}$$

which completes the proof of the lemma.  $\square$

**Lemma 3.2.** *The operator  $A(\xi) = -i\mathbf{P}_0(v \cdot \xi)\mathbf{P}_0 - \frac{i(v \cdot \xi)}{|\xi|^2}P_d$  generates a strongly continuous unitary group on  $N_0$  for any fixed  $|\xi| \neq 0$ , which satisfies for  $t > 0$  and  $f \in N_0 \cap L_\xi^2(\mathbb{R}_v^3)$  that*

$$\|e^{tA(\xi)}f\|_\xi = \|f\|_\xi. \quad (3.3)$$

*In addition, for any  $x \neq 0$  and  $f \in N_0 \cap L_\xi^2(\mathbb{R}_v^3)$ , it holds*

$$\int_{-\infty}^{+\infty} \|(x+iy)\mathbf{P}_0 - A(\xi)]^{-1}f\|_\xi^2 dy = \pi|x|^{-1}\|f\|_\xi^2. \quad (3.4)$$

*Proof.* Since the operator  $iA(\xi)$  is self-adjoint on  $N_0$  with respect to the inner product  $(\cdot, \cdot)_\xi$  defined by (2.29), we conclude by Lemma 6.3 that  $A(\xi)$  generates a strongly continuous unitary group on  $N_0$  and satisfies (3.3).

By a similar argument for proving (3.2), we can obtain for  $x > 0$

$$[(x+iy)\mathbf{P}_0 - A(\xi)]^{-1} = \int_0^\infty e^{-(x+iy)t} e^{tA(\xi)} dt,$$

from which we get for  $f \in N_0 \cap L_\xi^2(\mathbb{R}_v^3)$

$$\int_{-\infty}^{+\infty} \|(x+iy)\mathbf{P}_0 - A(\xi)]^{-1}f\|_\xi^2 dy = 2\pi \int_0^\infty e^{-2xt} \|e^{tA(\xi)}f\|_\xi^2 dt = 2\pi \int_0^\infty e^{-2xt} dt \|f\|_\xi^2.$$

As for  $x < 0$ , we have

$$[-(x+iy)\mathbf{P}_0 + A(\xi)]^{-1} = \int_0^\infty e^{(x+iy)t} e^{-tA(\xi)} dt.$$

This completes the proof of the lemma.  $\square$

By Lemma 2.3, we are able to show the invertibility of the operator  $I - K(\lambda - c(\xi))^{-1} + i(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}$  on  $L^2(\mathbb{R}^3)$  and estimate the bound of its norm by applying a similar arguments the one for Lemma 2.4. And we omit the proof for brevity. Indeed, we have

**Lemma 3.3.** *Given any constant  $r_0 > 0$ . Let  $\lambda = x + iy$  with  $x > -\alpha(r_0)$  and  $\alpha(r_0)$  defined in Lemma 2.4. Then, it holds*

$$\sup_{y \in \mathbb{R}, |\xi| \geq r_0} \|[I - K(\lambda - c(\xi))^{-1} + i(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}]^{-1}\| \leq C. \quad (3.5)$$

With the help of Lemmas 2.3–2.6 and Lemmas 3.1–3.3, we have the decomposition of the semigroup  $S(t, \xi) = e^{t\hat{B}(\xi)}$  given by

**Theorem 3.4.** *The semigroup  $S(t, \xi) = e^{t\hat{B}(\xi)}$  with  $\xi = s\omega \in \mathbb{R}^3$  and  $s = |\xi| \neq 0$  satisfies*

$$S(t, \xi)f = S_1(t, \xi)f + S_2(t, \xi)f, \quad f \in L_\xi^2(\mathbb{R}_v^3), \quad t > 0, \quad (3.6)$$

where

$$S_1(t, \xi)f = \sum_{j=-1}^3 e^{t\lambda_j(s)} (f, \overline{\psi_j(s, \omega)})_\xi \psi_j(s, \omega) 1_{\{|\xi| \leq r_0\}}, \quad (3.7)$$

with  $(\lambda_j(s), \psi_j(s, \omega))$  being the eigenvalue and eigenfunction of the operator  $\hat{B}(\xi)$  given by Theorem 2.11 for  $0 < |\xi| \leq r_0$ , and  $S_2(t, \xi)f =: S(t, \xi)f - S_1(t, \xi)f$  satisfies for a constant  $\sigma_0 > 0$  independent of  $\xi$  that

$$\|S_2(t, \xi)f\|_\xi \leq Ce^{-\sigma_0 t} \|f\|_\xi, \quad t \geq 0. \quad (3.8)$$

*Proof.* By Lemma 6.5, it is sufficient to prove the above decomposition for  $f \in D(\hat{B}(\xi)^2)$  because the domain  $D(\hat{B}(\xi)^2)$  is dense in  $L_\xi^2(\mathbb{R}_v^3)$ . By Lemma 6.4, the semigroup  $e^{t\hat{B}(\xi)}$  can be represented by

$$e^{t\hat{B}(\xi)}f = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} (\lambda - \hat{B}(\xi))^{-1} f d\lambda, \quad f \in D(\hat{B}(\xi)^2), \quad \kappa > 0. \quad (3.9)$$

It remains to analyze the resolvent  $(\lambda - \hat{B}(\xi))^{-1}$  for  $\xi \in \mathbb{R}^3$  in order to obtain the decomposition (3.6) for the semigroup  $e^{t\hat{B}(\xi)}$ .

First of all, we investigate the formula (3.9) for  $|\xi| \leq r_0$ . By (2.36) we have

$$(\lambda - \hat{B}(\xi))^{-1} = [(\lambda \mathbf{P}_0 - A(\xi))^{-1} \mathbf{P}_0 + (\lambda \mathbf{P}_1 - Q(\xi))^{-1} \mathbf{P}_1] - Z_1(\lambda, \xi), \quad (3.10)$$

with the operator  $Z_1(\lambda, \xi)$  defined by

$$Z_1(\lambda, \xi) = [(\lambda \mathbf{P}_0 - A(\xi))^{-1} \mathbf{P}_0 + (\lambda \mathbf{P}_1 - Q(\xi))^{-1} \mathbf{P}_1][I + Y_1(\lambda, \xi)]^{-1} Y_1(\lambda, \xi), \quad (3.11)$$

$$Y_1(\lambda, \xi) = i\mathbf{P}_1(v \cdot \xi) \mathbf{P}_0 (\lambda \mathbf{P}_0 - A(\xi))^{-1} \mathbf{P}_0 + i\mathbf{P}_0(v \cdot \xi) \mathbf{P}_1 (\lambda \mathbf{P}_1 - Q(\xi))^{-1} \mathbf{P}_1. \quad (3.12)$$

Substituting (3.10) into (3.9), we have the following decomposition of the semigroup  $e^{t\hat{B}(\xi)}$

$$e^{t\hat{B}(\xi)}f = e^{tQ(\xi)} \mathbf{P}_1 f - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} Z_2(\lambda, \xi) f d\lambda, \quad |\xi| \leq r_0, \quad (3.13)$$

with

$$Z_2(\lambda, \xi) = Z_1(\lambda, \xi) - (\lambda \mathbf{P}_0 - A(\xi))^{-1} \mathbf{P}_0.$$

To estimate the last term on the right hand side of (3.13), let us denote

$$U_{\kappa, N} = \frac{1}{2\pi i} \int_{-N}^N e^{(\kappa+iy)t} Z_2(\kappa+iy, \xi) f 1_{|\xi| \leq r_0} dy, \quad (3.14)$$

where the constant  $N > 0$  is chosen large enough so that  $N > y_1$  with  $y_1$  defined in Lemma 2.6. Since  $Z_2(\lambda, \xi)$  is analytic on the domain  $\text{Re } \lambda > -\mu/2$  with only finite singularities at  $\lambda = \lambda_j(s) \in \sigma(\hat{B}(\xi))$  for  $j = -1, 0, 1, 2$ , we can shift the integration (3.14) from the line  $\text{Re } \lambda = \kappa > 0$  to  $\text{Re } \lambda = -\mu/2$ . Then by the Residue Theorem, we obtain

$$U_{\kappa, N} = U_{-\frac{\mu}{2}, N} + H_N + 2\pi i \sum_{j=-1}^2 \text{Res} \{ e^{\lambda t} Z_2(\lambda, \xi) f; \lambda_j(s) \}, \quad (3.15)$$

where  $\text{Res}\{f(\lambda); \lambda_j\}$  means the residue of  $f$  at  $\lambda = \lambda_j$  and

$$H_N = \frac{1}{2\pi i} \left( \int_{-\frac{\mu}{2}+iN}^{\kappa+iN} - \int_{-\frac{\mu}{2}-iN}^{\kappa-iN} \right) e^{\lambda t} Z_2(\lambda, \xi) f 1_{|\xi| \leq r_0} d\lambda.$$

The right hand side of (3.15) is estimated as follows. By Lemma 2.5 it is easy to verify that

$$\|H_N\|_\xi \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (3.16)$$

Since  $(\lambda - A(\xi))^{-1}$  is analytic on the domain  $\{\lambda \in \mathbb{C} \mid \lambda \neq \alpha_j(\xi)\}$ , we have by Cauchy theorem and (2.25) that for any  $f \in N_0$ ,

$$\begin{aligned} & \left\| \int_{-\frac{\mu}{2}-iN}^{-\frac{\mu}{2}+iN} e^{\lambda t} (\lambda - A(\xi))^{-1} f d\lambda \right\|_\xi = \left\| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{(-\frac{\mu}{2}+Ne^{i\theta})t} \left(-\frac{\mu}{2} + Ne^{i\theta} - A(\xi)\right)^{-1} f i N e^{i\theta} d\theta \right\|_\xi \\ & \leq \frac{2}{N} \|f\|_\xi e^{-\frac{\mu t}{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tN \cos \theta} N d\theta \leq \frac{4}{N} \|f\|_\xi e^{-\frac{\mu t}{2}} \int_0^{\frac{N\pi}{2}} e^{-\frac{2}{\pi}ts} ds \rightarrow 0, \quad N \rightarrow \infty, \end{aligned}$$

which together with

$$\int_{-\frac{\mu}{2}-i\infty}^{-\frac{\mu}{2}+i\infty} e^{\lambda t} (\lambda - A(\xi))^{-1} f d\lambda = \lim_{N \rightarrow \infty} \int_{-\frac{\mu}{2}-iN}^{-\frac{\mu}{2}+iN} e^{\lambda t} (\lambda - A(\xi))^{-1} f d\lambda = 0,$$

give

$$\lim_{N \rightarrow \infty} U_{-\frac{\mu}{2}, N}(t) = U_{-\frac{\mu}{2}, \infty}(t) =: \int_{-\frac{\mu}{2}-i\infty}^{-\frac{\mu}{2}+i\infty} e^{\lambda t} Z_1(\lambda, \xi) f d\lambda. \quad (3.17)$$

Since it holds  $\|Y_1(-\frac{\mu}{2} + iy, \xi)\|_\xi \leq \frac{1}{2}$  for  $|\xi| \leq r_0$  with  $r_0 > 0$  being sufficiently small, the operator  $I - Y_1(-\frac{\mu}{2} + iy, \xi)$  is invertible on  $L_\xi^2(\mathbb{R}_v^3)$  and satisfies  $\sup_{|\xi| \leq r_0, y \in \mathbb{R}} \|[I - Y_1(-\frac{\mu}{2} + iy, \xi)]^{-1}\|_\xi \leq 2$ . Thus, we have for any  $f, g \in L_\xi^2(\mathbb{R}_v^3)$

$$\begin{aligned} |(U_{-\frac{\mu}{2}, \infty}(t)f, g)_\xi| &\leq e^{-\frac{\mu t}{2}} \int_{-\infty}^{+\infty} |(Z_1(-\frac{\mu}{2} + iy, \xi)f, g)_\xi| dy \\ &\leq C|\xi| e^{-\frac{\mu t}{2}} \int_{-\infty}^{+\infty} \left( \| [(-\frac{\mu}{2} + iy)\mathbf{P}_1 - Q(\xi)]^{-1} \mathbf{P}_1 f \| + \| [(-\frac{\mu}{2} + iy)\mathbf{P}_0 - A(\xi)]^{-1} \mathbf{P}_0 f \|_\xi \right) \\ &\quad \times \left( \| [(-\frac{\mu}{2} - iy)\mathbf{P}_1 - Q(-\xi)]^{-1} \mathbf{P}_1 g \| + \| [(-\frac{\mu}{2} - iy)\mathbf{P}_0 - A(-\xi)]^{-1} \mathbf{P}_0 g \|_\xi \right) dy. \end{aligned}$$

This together with (3.2) and (3.4) yield  $|(U_{-\frac{\mu}{2}, \infty}(t)f, g)_\xi| \leq Cr_0 \mu^{-1} e^{-\frac{\mu t}{2}} \|f\|_\xi \|g\|_\xi$ , and

$$\|U_{-\frac{\mu}{2}, \infty}(t)\|_\xi \leq Cr_0 \mu^{-1} e^{-\frac{\mu t}{2}}. \quad (3.18)$$

By  $\lambda_j(s) \in \rho(Q(\xi))$  and  $Z_2(\lambda, \xi) = (\lambda \mathbf{P}_1 - Q(\xi))^{-1} \mathbf{P}_1 - (\lambda - \hat{B}(\xi))^{-1}$ , we can prove

$$\text{Res}\{e^{\lambda t} Z_2(\lambda, \xi) f; \lambda_j(s)\} = -\text{Res}\{e^{\lambda t} (\lambda - \hat{B}(\xi))^{-1} f; \lambda_j(s)\}, \quad |\xi| \leq r_0, \quad (3.19)$$

and in particular

$$\text{Res}\{e^{\lambda t} (\lambda - \hat{B}(\xi))^{-1} f; \lambda_j(s)\} = e^{\lambda_j(s)t} P_j f = e^{\lambda_j(s)t} (f, \overline{\psi_j(s, \omega)})_\xi \psi_j(s, \omega), \quad j = -1, 0, 1, \quad (3.20)$$

$$\text{Res}\{e^{\lambda t} (\lambda - \hat{B}(\xi))^{-1} f; \lambda_2(s)\} = e^{\lambda_2(s)t} P_2 f = e^{\lambda_2(s)t} \sum_{j=2,3} (f, \overline{\psi_j(s, \omega)})_\xi \psi_j(s, \omega). \quad (3.21)$$

Indeed, by the spectral representation formula in [10], we have for  $|\xi| \leq r_0$  and  $|\lambda - \lambda_j(s)| \leq \delta$  with  $\delta > 0$  being small

$$(\lambda - \hat{B}(\xi))^{-1} = \sum_{j=-1}^2 \left[ \frac{P_j}{\lambda - \lambda_j(s)} + \sum_{m=1}^{n_j} \frac{D_j^m}{(\lambda - \lambda_j(s))^{m+1}} \right] + S(\lambda),$$

where  $P_j$  is the projection operator associated with  $\lambda_j(s)$ ,  $D_j$  is the nilpotent operator associated with  $\lambda_j(s)$ , and the operator  $S(\lambda)$  is holomorphic on the domain  $\{\lambda \mid |\lambda - \lambda_j(s)| < \delta\}$  with  $\delta > 0$  being small enough.

We claim that  $D_j = 0$  for all  $j$ ,  $-1 \leq j \leq 2$ . Indeed, if  $D_j \neq 0$ , then there exists  $n_j$  such that  $D_j^{n_j} \neq 0$  for  $m \leq n_j$  and  $D_j^{n_j+1} = 0$ . Thus  $D_j^{n_j} = (\hat{B}(\xi) - \lambda_j(\xi))^{n_j} P_j \neq 0$ ,  $D_j^{n_j+1} = (\hat{B}(\xi) - \lambda_j(\xi))^{n_j+1} P_j = 0$ . Assume that  $n_j \geq 1$  and let  $x \in L_\xi^2(\mathbb{R}_v^3)$  be such that  $y = [\hat{B}(\xi) - \lambda_j(\xi)]^{n_j} P_j x \neq 0$ . Then  $[\hat{B}(\xi) - \lambda_j(\xi)]y = 0$ . Hence  $y = C(s)\psi_j(s, \omega)$  for some constant  $C(s) \neq 0$ . We may normalize  $x$  so that  $C(s) = 1$ . Taking the inner product with  $\overline{\psi_j(s, \omega)}$ , we have

$$\begin{aligned} 1 &= (\psi_j(s, \omega), \overline{\psi_j(s, \omega)})_\xi = ([\hat{B}(\xi) - \lambda_j(s)]^{n_j} P_j x, \overline{\psi_j(s, \omega)})_\xi \\ &= ([\hat{B}(\xi) - \lambda_j(s)]^{n_j-1} P_j x, [\hat{B}(-\xi) - \overline{\lambda_j(s)}] \overline{\psi_j(s, \omega)})_\xi = 0, \end{aligned}$$

because  $\overline{\psi_j(s, \omega)}$  is the eigenfunction of  $\hat{B}(-\xi)$  with eigenvalue  $\overline{\lambda_j(s)}$ . This is a contradiction. Thus it holds  $D_j = 0$ , and then we have (3.19)–(3.21) by Cauchy's theorem.

Therefore, we conclude from (3.13) and (3.14)-(3.21) that

$$e^{t\hat{B}(\xi)}f = e^{tQ(\xi)}\mathbf{P}_1 f 1_{\{|\xi| \leq r_0\}} + U_{-\frac{\mu}{2}, \infty}(t) + \sum_{j=-1}^3 e^{t\lambda_j(s)}(f, \overline{\psi_j(s, \omega)})_{\xi} \psi_j(s, \omega) 1_{\{|\xi| \leq r_0\}}, \quad |\xi| \leq r_0. \quad (3.22)$$

Next, we turn to prove the formula (3.9) for  $|\xi| > r_0$ . By (2.19) we have

$$(\lambda - \hat{B}(\xi))^{-1} = (\lambda - c(\xi))^{-1} + Z_3(\lambda, \xi), \quad (3.23)$$

with the operator  $Z_3(\lambda, \xi)$  defined by

$$Z_3(\lambda, \xi) = (\lambda - c(\xi))^{-1}[I - Y_2(\lambda, \xi)]^{-1}Y_2(\lambda, \xi), \quad (3.24)$$

$$Y_2(\lambda, \xi) =: (K - i(v \cdot \xi)|\xi|^{-2}P_d)(\lambda - c(\xi))^{-1}. \quad (3.25)$$

Substituting (3.23) into (3.9) yields

$$e^{t\hat{B}(\xi)}f = e^{tc(\xi)}f + \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} e^{\lambda t} Z_3(\lambda, \xi) f d\lambda, \quad |\xi| > r_0. \quad (3.26)$$

Similarly, in order to estimate the last term on the right hand side of (3.26), let us denote

$$V_{\kappa, N} = \frac{1}{2\pi i} \int_{-N}^N e^{(\kappa + iy)t} Z_3(\kappa + iy, \xi) 1_{|\xi| > r_0} dy \quad (3.27)$$

for sufficiently large constant  $N > 0$  as in (3.14). Since the operator  $Z_3(\lambda, \xi)$  is analytic on the domain  $\operatorname{Re} \lambda \geq -\sigma_0$  for the constant  $\sigma_0 = \frac{1}{2}\alpha(r_0)$  with  $\alpha(r_0) > 0$  given by Lemma 2.4, we can again shift the integration of (3.27) from the line  $\operatorname{Re} \lambda = \kappa > 0$  to  $\operatorname{Re} \lambda = -\sigma_0$  to obtain

$$V_{\kappa, N} = V_{-\sigma_0, N} + I_N, \quad (3.28)$$

with

$$I_N = \frac{1}{2\pi i} \left( \int_{-\sigma_0 + iN}^{-\kappa + iN} - \int_{-\sigma_0 - iN}^{-\kappa - iN} \right) e^{\lambda t} Z_3(\lambda, \xi) f 1_{|\xi| > r_0} d\lambda.$$

By Lemma 2.3 and Lemma 3.3, it is easy to verify

$$\|I_N\| \rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad \sup_{|\xi| > r_0, y \in \mathbb{R}} \|[I - Y_2(-\sigma_0 + iy, \xi)]^{-1}\| \leq C. \quad (3.29)$$

Define

$$V_{-\sigma_0, \infty}(t) = \lim_{N \rightarrow \infty} V_{-\sigma_0, N}(t) = \int_{-\infty}^{+\infty} e^{(-\sigma_0 + iy)t} Z_3(-\sigma_0 + iy, \xi) f dy. \quad (3.30)$$

We have for any  $f, g \in L_{\xi}^2(\mathbb{R}_v^3)$

$$\begin{aligned} |(V_{-\sigma_0, \infty}(t)f, g)| &\leq C e^{-\sigma_0 t} \int_{-\infty}^{+\infty} |(Z_3(-\sigma_0 + iy, \xi)f, g)| dy \\ &\leq C(\|K\| + r_0^{-1}) e^{-\sigma_0 t} \int_{-\infty}^{+\infty} \|(-\sigma_0 + iy - c(\xi))^{-1} f\| \|(-\sigma_0 - iy - c(-\xi))^{-1} g\| dy \\ &\leq C(\|K\| + r_0^{-1}) e^{-\sigma_0 t} (\nu_0 - \sigma_0)^{-1} \|f\| \|g\|, \end{aligned} \quad (3.31)$$

where we have used the fact (cf. Lemma 2.2.13 of [19])

$$\int_{-\infty}^{+\infty} \|(x + iy - c(\xi))^{-1} f\|^2 dy \leq \pi(x + \nu_0)^{-1} \|f\|^2, \quad x > -\nu_0.$$

From (3.31) and the fact  $\|f\|^2 \leq \|f\|_{\xi}^2 \leq (1 + r_0^{-2}) \|f\|^2$  for  $|\xi| > r_0$ , we have

$$\|V_{-\sigma_0, \infty}(t)\|_{\xi} \leq C e^{-\sigma_0 t} (\nu_0 - \sigma_0)^{-1}. \quad (3.32)$$

Therefore, we conclude from (3.26) and (3.27)–(3.30) that

$$e^{t\hat{B}(\xi)}f = e^{tc(\xi)}f1_{\{|\xi|>r_0\}} + V_{-\sigma_0,\infty}(t), \quad |\xi| > r_0. \quad (3.33)$$

The combination of (3.22) and (3.33) gives rise to (3.6) with  $S_1(t, \xi)f$  and  $S_2(t, \xi)f$  defined by

$$\begin{aligned} S_1(t, \xi)f &= \sum_{j=-1}^3 e^{t\lambda_j(s)}(f, \overline{\psi_j(s, \omega)})_{\xi} \psi_j(s, \omega) 1_{\{|\xi| \leq r_0\}}, \\ S_2(t, \xi)f &= \left( e^{tQ(\xi)} \mathbf{P}_1 f + U_{-\frac{\mu}{2}, \infty}(t) \right) 1_{\{|\xi| \leq r_0\}} + \left( e^{tc(\xi)} f + V_{-\sigma_0, \infty}(t) \right) 1_{\{|\xi| > r_0\}}. \end{aligned}$$

In particular,  $S_2(t, \xi)f$  satisfies (3.8) in terms of (3.1), (3.18), (3.32) and the estimate  $\|e^{tc(\xi)}1_{\{|\xi|>r_0\}}\|_{\xi} \leq Ce^{-\nu_0 t}$  because (2.8) and (1.13).  $\square$

### 3.2 Optimal time-decay rates of linearized VPB

Let us introduce a Sobolev space of function  $f = f(x, v)$  by  $H_P^l = \{f \in L^2(\mathbb{R}_{x,v}^6) \mid \|f\|_{H_P^l} < \infty\}$  ( $L_P^2 = H_P^0$ ) with the norm  $\|\cdot\|_{H_P^l}$  defined by

$$\begin{aligned} \|f\|_{H_P^l} &= \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^l \|\hat{f}\|_{\xi}^2 d\xi \right)^{1/2} \\ &= \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^l \left( \int_{\mathbb{R}^3} |\hat{f}|^2 dv + \frac{1}{|\xi|^2} \left| \int_{\mathbb{R}^3} \hat{f} \sqrt{M} dv \right|^2 \right) d\xi \right)^{1/2}, \end{aligned}$$

where  $\hat{f} = \hat{f}(\xi, v)$  denotes the Fourier transform of  $f(x, v)$  with respect to the spatial variable  $x \in \mathbb{R}^3$ . Note that it holds

$$\|f\|_{H_P^l}^2 = \|f\|_{L^2(\mathbb{R}_v^3, H^l(\mathbb{R}_x^3))}^2 + \|\nabla_x \Delta_x^{-1}(f, \sqrt{M})\|_{H^l(\mathbb{R}_x^3)}^2.$$

Denote

$$L^{2,q} = L^2(\mathbb{R}_v^3, L^q(\mathbb{R}_x^3)), \quad \|f\|_{L^{2,q}} = \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |f(x, v)|^q dx \right)^{2/q} dv \right)^{1/2}.$$

For any  $f_0 \in L^2(\mathbb{R}_{x,v}^6)$ , set

$$e^{tB}f_0 = (\mathcal{F}^{-1}e^{t\hat{B}(\xi)}\mathcal{F})f_0. \quad (3.34)$$

By Lemma 2.1, it holds

$$\|e^{tB}f_0\|_{H_P^l} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^l \|e^{t\hat{B}(\xi)}\hat{f}_0\|_{\xi}^2 d\xi \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^l \|\hat{f}_0\|_{\xi}^2 d\xi = \|f_0\|_{H_P^l}.$$

This means that the linear operator  $B$  generates a strongly continuous contraction semigroup  $e^{tB}$  in  $H_P^l$ , and therefore,  $f(x, v, t) = e^{tB}f_0(x, v)$  is a global solution to the IVP (2.1) for the linearized Vlasov-Poisson-Boltzmann equation for any  $f_0 \in H_P^l$ . We are now going to establish the time-decay rates of the global solution.

First of all, we have the bounds of the time decay rates of the global solutions as follows.

**Theorem 3.5.** *Set  $\nabla_x \Phi(t) = \nabla_x \Delta_x^{-1}(e^{tB}f_0, \sqrt{M})$ . If  $f_0 \in L^2(\mathbb{R}_v^3; H^l(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3))$ , then it holds*

$$\|(\partial_x^\alpha e^{tB}f_0, \chi_0)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-(\frac{3}{4} + \frac{|\alpha|}{2})} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad (3.35)$$

$$\|(\partial_x^\alpha e^{tB}f_0, \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-(\frac{1}{4} + \frac{|\alpha|}{2})} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad j = 1, 2, 3, \quad (3.36)$$

$$\|(\partial_x^\alpha e^{tB}f_0, \chi_4)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-(\frac{3}{4} + \frac{|\alpha|}{2})} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad (3.37)$$

$$\|\partial_x^\alpha \nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-(\frac{1}{4} + \frac{|\alpha|}{2})} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad (3.38)$$

$$\|\mathbf{P}_1(\partial_x^\alpha e^{tB}f_0)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C(1+t)^{-(\frac{3}{4} + \frac{|\alpha|}{2})} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}). \quad (3.39)$$

where  $\chi_j$ ,  $j = 0, 1, 2, 3, 4$ , is defined by (1.11) and  $|\alpha| \leq l$ . Moreover, if  $f_0 \in L^2(\mathbb{R}_v^3; H^l(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3))$  and  $(f_0, \chi_0) = 0$ , then it holds

$$\|(\partial_x^\alpha e^{tB} f_0, \chi_0)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\left(\frac{5}{4} + \frac{|\alpha|}{2}\right)} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad (3.40)$$

$$\|(\partial_x^\alpha e^{tB} f_0, \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\left(\frac{3}{4} + \frac{|\alpha|}{2}\right)} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad j = 1, 2, 3, \quad (3.41)$$

$$\|(\partial_x^\alpha e^{tB} f_0, \chi_4)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\left(\frac{3}{4} + \frac{|\alpha|}{2}\right)} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad (3.42)$$

$$\|\partial_x^\alpha \nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\left(\frac{3}{4} + \frac{|\alpha|}{2}\right)} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}), \quad (3.43)$$

$$\|\mathbf{P}_1(\partial_x^\alpha e^{tB} f_0)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C(1+t)^{-\left(\frac{5}{4} + \frac{|\alpha|}{2}\right)} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}). \quad (3.44)$$

*Proof.* We prove (3.35)–(3.39) first. It follows from Theorem 3.4 and the Planchel's equality that

$$\begin{aligned} \|\partial_x^\alpha (e^{tB} f_0, \chi_j)\|_{L^2(\mathbb{R}_x^3)} &= \|\xi^\alpha (S(t, \xi) \hat{f}_0, \chi_j)\|_{L^2(\mathbb{R}_\xi^3)} \\ &\leq \|\xi^\alpha (S_1(t, \xi) \hat{f}_0, \chi_j)\|_{L^2(\mathbb{R}_\xi^3)} + \|\xi^\alpha (S_2(t, \xi) \hat{f}_0, \chi_j)\|_{L^2(\mathbb{R}_\xi^3)} \\ &\leq \|\xi^\alpha (S_1(t, \xi) \hat{f}_0, \chi_j)\|_{L^2(\mathbb{R}_\xi^3)} + \|\xi^\alpha S_2(t, \xi) \hat{f}_0\|_{L^2(\mathbb{R}_{\xi,v}^6)}, \end{aligned} \quad (3.45)$$

$$\|\partial_x^\alpha \nabla_x \Phi\|_{L^2(\mathbb{R}_x^3)} \leq \|\xi^\alpha |\xi|^{-1} (S_1(t, \xi) \hat{f}_0, \sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)} + \|\xi^\alpha |\xi|^{-1} (S_2(t, \xi) \hat{f}_0, \sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}. \quad (3.46)$$

By (3.8) and

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{(\xi^\alpha)^2}{|\xi|^2} |(\hat{f}_0, \sqrt{M})|^2 d\xi &\leq \sup_{|\xi| \leq 1} |(\hat{f}_0, \sqrt{M})|^2 \int_{|\xi| \leq 1} \frac{1}{|\xi|^2} d\xi + \int_{|\xi| > 1} (\xi^\alpha)^2 |(\hat{f}_0, \sqrt{M})|^2 d\xi \\ &\leq C(\|(\hat{f}_0, \sqrt{M})\|_{L^1(\mathbb{R}_x^3)}^2 + \|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)}^2), \end{aligned}$$

we can estimate the high frequency terms on the right hand side of (3.45)–(3.46) as follows:

$$\begin{aligned} &\|\xi^\alpha S_2(t, \xi) \hat{f}_0\|_{L^2(\mathbb{R}_{\xi,v}^6)}^2 + \|\xi^\alpha |\xi|^{-1} (S_2(t, \xi) \hat{f}_0, \sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 \\ &= \int_{\mathbb{R}^3} (\xi^\alpha)^2 (\|S_2(t, \xi) \hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 + \frac{1}{|\xi|^2} |(S_2(t, \xi) \hat{f}_0, \sqrt{M})|^2) d\xi \\ &\leq C \int_{\mathbb{R}^3} e^{-2\sigma_0 t} (\xi^\alpha)^2 (\|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 + \frac{1}{|\xi|^2} |(\hat{f}_0, \sqrt{M})|^2) d\xi \\ &\leq C e^{-2\sigma_0 t} (\|(f_0, \sqrt{M})\|_{L^1(\mathbb{R}_x^3)}^2 + \|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)}^2). \end{aligned} \quad (3.47)$$

By (3.7), we have for  $|\xi| \leq r_0$  that

$$S_1(t, \xi) \hat{f}_0 = \sum_{j=-1}^3 e^{t\lambda_j(|\xi|)} P_j(\xi) \hat{f}_0, \quad \xi = s\omega, \quad s = |\xi| \neq 0,$$

where

$$P_j(\xi) \hat{f}_0 = (\hat{f}_0, \overline{\psi_j(s, \omega)})_\xi \psi_j(s, \omega) \mathbf{1}_{\{|\xi| \leq r_0\}}.$$

According to (2.69) and (2.70), we can decompose  $P_j(\xi) \hat{f}$  for  $|\xi| \leq r_0$  as

$$P_j(\xi) \hat{f}_0 = (\hat{n}_0 \cdot W^j)(W^j \cdot v) \sqrt{M} + |\xi| T_j(\xi) \hat{f}_0, \quad j = 2, 3, \quad (3.48)$$

$$P_0(\xi) \hat{f}_0 = \left(\hat{q}_0 - \sqrt{\frac{2}{3}} \hat{n}_0\right) \chi_4 + |\xi| T_0(\xi) \hat{f}_0, \quad (3.49)$$

$$\begin{aligned} P_{\pm 1}(\xi) \hat{f}_0 &= \frac{1}{2} \left[ (\hat{n}_0 \cdot \omega) \mp \frac{1}{|\xi|} \hat{n}_0 \right] (v \cdot \omega) \sqrt{M} + \frac{1}{2} \hat{n}_0 \left( \sqrt{M} + \sqrt{\frac{2}{3}} \chi_4 \right) \\ &\quad \mp \frac{i}{2} \hat{n}_0 (L \mp i \mathbf{P}_1)^{-1} \mathbf{P}_1 (v \cdot \omega)^2 \sqrt{M} + |\xi| T_{\pm 1}(\xi) \hat{f}_0, \end{aligned} \quad (3.50)$$

where  $(\hat{n}_0, \hat{m}_0, \hat{q}_0)$  is the Fourier transform of the macroscopic density, momentum and energy  $(n_0, m_0, q_0)$  of the initial data  $f_0 \in L^2(\mathbb{R}_{x,v}^6)$  defined by

$$(n_0, m_0, q_0) =: ((f_0, \chi_0), (f_0, v\sqrt{M}), (f_0, \chi_4)), \quad (3.51)$$

$W^j$  is given by (2.70), and  $T_j(\xi)$ ,  $-1 \leq j \leq 3$ , is the linear operators with the norm  $\|T_j(\xi)\|$  being uniformly bounded for  $|\xi| \leq r_0$ .

Since  $(T_j(\xi)\hat{f}_0, \sqrt{M}) = (T_j(\xi)\hat{f}_0, \chi_4) = 0$  for  $j = 2, 3$ , the macroscopic density, momentum and energy of  $S_1(t, \xi)\hat{f}_0$  satisfy

$$(S_1(t, \xi)\hat{f}_0, \sqrt{M}) = \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} \hat{n}_0 + |\xi| \sum_{j=-1}^1 e^{\lambda_j(|\xi|)t} (T_j(\xi)\hat{f}_0, \sqrt{M}), \quad (3.52)$$

$$\begin{aligned} (S_1(t, \xi)\hat{f}_0, v\sqrt{M}) &= \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} \left[ (\hat{m}_0 \cdot \omega) - \frac{j}{|\xi|} \hat{n}_0 \right] \omega + \sum_{j=2,3} e^{\lambda_j(|\xi|)t} (\hat{m}_0 \cdot W^j) W^j \\ &\quad + |\xi| \sum_{j=-1}^3 e^{\lambda_j(|\xi|)t} (T_j(\xi)\hat{f}_0, v\sqrt{M}), \end{aligned} \quad (3.53)$$

$$(S_1(t, \xi)\hat{f}_0, \chi_4) = \sqrt{\frac{1}{6}} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} \hat{n}_0 + e^{\lambda_0(|\xi|)t} (\hat{q}_0 - \sqrt{\frac{2}{3}} \hat{n}_0) + |\xi| \sum_{j=-1}^1 e^{\lambda_j(|\xi|)t} (T_j(\xi)\hat{f}_0, \chi_4). \quad (3.54)$$

Note that

$$\operatorname{Re} \lambda_j(|\xi|) = a_j |\xi|^2 (1 + O(|\xi|)) \leq -\beta |\xi|^2, \quad |\xi| \leq r_0, \quad (3.55)$$

where  $\beta > 0$  denotes a generic constant that will also be used later. We obtain by (3.52)–(3.54) that

$$\begin{aligned} \|\xi^\alpha (S_1(t, \xi)\hat{f}_0, \sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 &\leq C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 e^{-2\beta|\xi|^2 t} \left( |\hat{n}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 \right) d\xi \\ &\leq C(1+t)^{-(3/2+|\alpha|)} \left( \|n_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|f_0\|_{L^{2,1}}^2 \right), \end{aligned} \quad (3.56)$$

$$\begin{aligned} \|\xi^\alpha (S_1(t, \xi)\hat{f}_0, v\sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 &\leq C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 e^{-2\beta|\xi|^2 t} \left( |\hat{m}_0|^2 + |\xi|^{-2} |\hat{n}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 \right) d\xi \\ &\leq C(1+t)^{-(1/2+|\alpha|)} \left( \|n_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|m_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|f_0\|_{L^{2,1}}^2 \right), \end{aligned} \quad (3.57)$$

$$\begin{aligned} \|\xi^\alpha (S_1(t, \xi)\hat{f}_0, \chi_4)\|_{L^2(\mathbb{R}_\xi^3)}^2 &\leq C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 e^{-2\beta|\xi|^2 t} \left( |\hat{q}_0|^2 + |\hat{n}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 \right) d\xi \\ &\leq C(1+t)^{-(3/2+|\alpha|)} \left( \|n_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|q_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|f_0\|_{L^{2,1}}^2 \right). \end{aligned} \quad (3.58)$$

Since

$$\|(n_0, m_0, q_0)\|_{L^1(\mathbb{R}_x^3)} \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f_0| dx (\sqrt{M}, v\sqrt{M}, \chi_4) dv \leq \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |f_0| dx \right)^2 dv \right)^{1/2},$$

we obtain from (3.56)–(3.58) that

$$\begin{aligned} \|\xi^\alpha (S_1(t, \xi)\hat{f}_0, \sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 &\leq C(1+t)^{-(3/2+|\alpha|)} \|f_0\|_{L^{2,1}}, \\ \|\xi^\alpha (S_1(t, \xi)\hat{f}_0, v\sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 &\leq C(1+t)^{-(1/2+|\alpha|)} \|f_0\|_{L^{2,1}}, \\ \|\xi^\alpha (S_1(t, \xi)\hat{f}_0, \chi_4)\|_{L^2(\mathbb{R}_\xi^3)}^2 &\leq C(1+t)^{-(3/2+|\alpha|)} \|f_0\|_{L^{2,1}}, \end{aligned}$$

which together with (3.45), (3.46) and (3.47) lead to (3.35)–(3.38).

By (3.48)–(3.50), we can represent the corresponding microscopic part of  $S_1(t, \xi)\hat{f}_0$  by

$$\mathbf{P}_1(S_1(t, \xi)\hat{f}_0) = \sum_{j=-1}^3 e^{t\lambda_j(|\xi|)} \mathbf{P}_1(P_j(\xi)\hat{f}_0)$$



$$= |\xi| \sum_{j=-1}^3 e^{t\lambda_j(|\xi|)} \mathbf{P}_1(T_j(\xi)\hat{f}_0) - \frac{1}{2} \sum_{j=\pm 1} e^{t\lambda_j(|\xi|)} \hat{n}_0 j i(L - j i \mathbf{P}_1)^{-1} \mathbf{P}_1(v \cdot \omega)^2 \sqrt{M}, \quad (3.59)$$

to obtain

$$\begin{aligned} \|\xi^\alpha \mathbf{P}_1(S_1(t, \xi)\hat{f}_0)\|^2 &\leq C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 e^{-2\beta|\xi|^2 t} |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 d\xi + C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 e^{-2\beta|\xi|^2 t} |\hat{n}_0|^2 d\xi \\ &\leq C(1+t)^{-(3/2+|\alpha|)} (\|f_0\|_{L^{2,1}}^2 + \|n_0\|_{L^1(\mathbb{R}_x^3)}^2). \end{aligned} \quad (3.60)$$

This and (3.47) give (3.39).

Similarly, we can obtain the time decay rates (3.40)–(3.44) from (3.56)–(3.60) for the case  $\hat{n}_0 = 0$  (which is true if  $(f_0, \chi_0) = 0$ ).  $\square$

Then, we show that the above time-decay rates of the global solutions are optimal. Indeed, we have

**Theorem 3.6.** *Assume that  $f_0 \in L^2(\mathbb{R}_v^3; H^l(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3))$  and satisfies that  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| \geq d_0 > 0$  and  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)|$  with  $d_0 > 0$  and  $d_1 > 0$  being two constants, and  $\chi_j$ ,  $j = 0, 1, 2, 3, 4$ , defined by (1.11), then it holds for  $t > 0$  being large enough that*

$$C_1(1+t)^{-\frac{3}{4}} \leq \|e^{tB} f_0, \chi_0\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{3}{4}}, \quad (3.61)$$

$$C_1(1+t)^{-\frac{1}{4}} \leq \|e^{tB} f_0, \chi_j\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{1}{4}}, \quad j = 1, 2, 3, \quad (3.62)$$

$$C_1(1+t)^{-\frac{3}{4}} \leq \|e^{tB} f_0, \chi_4\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{3}{4}}, \quad (3.63)$$

$$C_1(1+t)^{-\frac{1}{4}} \leq \|\nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{1}{4}}, \quad (3.64)$$

$$C_1(1+t)^{-\frac{3}{4}} \leq \|\mathbf{P}_1(e^{tB} f_0)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C_2(1+t)^{-\frac{3}{4}}, \quad (3.65)$$

where  $C_2 \geq C_1 > 0$  are two generic constants. In addition, we have

$$C_1(1+t)^{-\frac{1}{4}} \leq \|e^{tB} f_0\|_{H_v^1} \leq C_2(1+t)^{-\frac{1}{4}}. \quad (3.66)$$

If  $(f_0, \chi_0) = 0$ ,  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, (v \cdot \omega)\sqrt{M})| \geq d_0$  with  $\omega = \frac{\xi}{|\xi|} \in \mathbb{S}^2$  and  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_0$ , then it holds for  $t > 0$  being large enough that

$$C_1(1+t)^{-\frac{5}{4}} \leq \|e^{tB} f_0, \chi_0\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{5}{4}}, \quad (3.67)$$

$$C_1(1+t)^{-\frac{3}{4}} \leq \|e^{tB} f_0, \chi_j\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{3}{4}}, \quad j = 1, 2, 3, \quad (3.68)$$

$$C_1(1+t)^{-\frac{3}{4}} \leq \|e^{tB} f_0, \chi_4\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{3}{4}}, \quad (3.69)$$

$$C_1(1+t)^{-\frac{3}{4}} \leq \|\nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C_2(1+t)^{-\frac{3}{4}}, \quad (3.70)$$

$$C_1(1+t)^{-\frac{5}{4}} \leq \|\mathbf{P}_1(e^{tB} f_0)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C_2(1+t)^{-\frac{5}{4}}. \quad (3.71)$$

In particular, we have

$$C_1(1+t)^{-\frac{3}{4}} \leq \|e^{tB} f_0\|_{H_v^1} \leq C_2(1+t)^{-\frac{3}{4}}. \quad (3.72)$$

*Proof.* By Theorem 3.5, we only need to show the lower bounds of the time-decay rates for the solution  $e^{tB} f_0$  under the assumptions of Theorem 3.6. Let us prove (3.61)–(3.65) first. Indeed, in terms of Theorem 3.4, we can verify that

$$\begin{aligned} \|(e^{tB} f, \chi_j)\|_{L^2(\mathbb{R}_x^3)} &= \|(S_1(t, \xi)\hat{f}_0, \chi_j) + (S_2(t, \xi)\hat{f}_0, \chi_j)\|_{L^2(\mathbb{R}_x^3)} \\ &\geq \|(S_1(t, \xi)\hat{f}_0, \chi_j)\|_{L^2(\mathbb{R}_x^3)} - \|S_2(t, \xi)\hat{f}_0\|_{L^2(\mathbb{R}_{\xi,v}^6)} \\ &\geq \|(S_1(t, \xi)\hat{f}_0, \chi_j)\|_{L^2(\mathbb{R}_x^3)} - C e^{-\sigma_0 t} (\|f_0\|_{L^{2,1}} + \|f_0\|_{L^2(\mathbb{R}_{x,v}^6)}), \quad j = 0, 1, 2, 3, 4, \end{aligned} \quad (3.73)$$

$$\begin{aligned} \|\nabla_x \Phi\|_{L^2(\mathbb{R}_x^3)} &\geq \|\xi|^{-1} (S_1(t, \xi)\hat{f}_0, \chi_0)\|_{L^2(\mathbb{R}_x^3)} - \|\xi|^{-1} (S_2(t, \xi)\hat{f}_0, \chi_0)\|_{L^2(\mathbb{R}_x^3)} \\ &\geq \|\xi|^{-1} (S_1(t, \xi)\hat{f}_0, \chi_0)\|_{L^2(\mathbb{R}_x^3)} - C e^{-\sigma_0 t} (\|f_0\|_{L^{2,1}} + \|f_0\|_{L^2(\mathbb{R}_{x,v}^6)}), \end{aligned} \quad (3.74)$$

$$\begin{aligned}
\|\mathbf{P}_1(e^{tB}f)\|_{L^2(\mathbb{R}_{x,v}^6)} &\geq \|\mathbf{P}_1(S_1(t,\xi)\hat{f}_0)\|_{L^2(\mathbb{R}_{\xi,v}^6)} - \|\mathbf{P}_1(S_2(t,\xi)\hat{f}_0)\|_{L^2(\mathbb{R}_{\xi,v}^6)} \\
&\geq \|\mathbf{P}_1(S_1(t,\xi)\hat{f}_0)\|_{L^2(\mathbb{R}_{\xi,v}^6)} - Ce^{-\sigma_0 t}(\|f_0\|_{L^{2,1}} + \|f_0\|_{L^2(\mathbb{R}_{x,v}^6)}),
\end{aligned} \tag{3.75}$$

where we have used (3.47) for  $\alpha = 0$ , namely,

$$\int_{\mathbb{R}^3} (\|S_2(t,\xi)\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 + |\xi|^{-2}|(S_2(t,\xi)\hat{f}_0, \sqrt{M})|^2) d\xi \leq Ce^{-\sigma_0 t}(\|f_0\|_{L^{2,1}}^2 + \|f_0\|_{L^2(\mathbb{R}_{x,v}^6)}^2).$$

By (3.52) and  $\lambda_{-1}(|\xi|) = \overline{\lambda_1(|\xi|)}$ , we have

$$\begin{aligned}
|(S_1(t,\xi)\hat{f}_0, \sqrt{M})|^2 &= \left| e^{\operatorname{Re}\lambda_1(|\xi|)t} \cos(\operatorname{Im}\lambda_1(|\xi|)t)\hat{n}_0 + |\xi| \sum_{j=-1}^1 e^{\lambda_j(|\xi|)t} (T_j(\xi)\hat{f}_0, \sqrt{M}) \right|^2 \\
&\geq \frac{1}{2} e^{2\operatorname{Re}\lambda_1(|\xi|)t} \cos^2(\operatorname{Im}\lambda_1(|\xi|)t) |\hat{n}_0|^2 - C|\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2.
\end{aligned} \tag{3.76}$$

Since

$$\cos^2(\operatorname{Im}\lambda_1(|\xi|)t) \geq \frac{1}{2} \cos^2[(1+b_1|\xi|^2)t] - O(|\xi|^3 t^2),$$

and

$$\operatorname{Re}\lambda_j(|\xi|) = a_j|\xi|^2(1+O(|\xi|)) \geq -\eta|\xi|^2, \quad |\xi| \leq r_0,$$

for some constant  $\eta > 0$ , we obtain by (3.76) that

$$\begin{aligned}
\|(S_1(t,\xi)\hat{f}_0, \sqrt{M})\|_{L_\xi^2}^2 &\geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2 t} \cos^2(t+b_1|\xi|^2 t) d\xi \\
&\quad - C \int_{|\xi| \leq r_0} e^{-2\beta|\xi|^2 t} [(|\xi|^3 t)^2 |\hat{n}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2] d\xi \\
&\geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2 t} \cos^2(t+b_1|\xi|^2 t) d\xi - C(\|n_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|f_0\|_{L^{2,1}}^2)(1+t)^{-5/2} \\
&=: I_1 - C(1+t)^{-5/2}.
\end{aligned} \tag{3.77}$$

Since it holds for  $t \geq t_0 =: \frac{L^2}{r_0^2}$  with the constant  $L \geq \sqrt{\frac{4\pi}{b_1}}$  that

$$\begin{aligned}
I_1 &= \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2 t} \cos^2[(1+b_1|\xi|^2)t] d\xi \\
&= \frac{d_0^2}{4} t^{-3/2} \int_{|\zeta| \leq r_0 \sqrt{t}} e^{-2\eta|\zeta|^2} \cos^2(t+b_1|\zeta|^2) d\zeta \geq \pi d_0^2 (1+t)^{-3/2} \int_0^L r^2 e^{-2\eta r^2} \cos^2(t+b_1 r^2) dr \\
&\geq (1+t)^{-3/2} \frac{\pi d_0^2 L}{2} e^{-2\eta L^2} \int_{L/2}^L r \cos^2(t+b_1 r^2) dr = (1+t)^{-3/2} \frac{\pi d_0^2 L}{4b_1} e^{-2\eta L^2} \int_{t+\frac{b_1 L^2}{4}}^{t+b_1 L^2} \cos^2 y dy \\
&\geq (1+t)^{-3/2} \frac{\pi d_0^2 L}{4b_1} e^{-2\eta L^2} \int_0^\pi \cos^2 y dy \geq C_3 (1+t)^{-3/2},
\end{aligned} \tag{3.78}$$

where  $C_3 > 0$  denotes a generic positive constant. We can substitute (3.77) and (3.78) into (3.73) with  $j = 0$  to prove (3.61).

By (3.53), we can decompose the macroscopic momentum into

$$(S_1(t,\xi)\hat{f}_0, v\sqrt{M}) = -\frac{i}{|\xi|} e^{\operatorname{Re}\lambda_1(|\xi|)t} \sin(\operatorname{Im}\lambda_1(|\xi|)t) \hat{n}_0 + T_5(t,\xi)\hat{f}_0,$$

with  $T_5(t,\xi)\hat{f}_0 =: (S_1(t,\xi)\hat{f}_0, v\sqrt{M}) + \frac{i}{|\xi|} e^{\operatorname{Re}\lambda_1(|\xi|)t} \sin(\operatorname{Im}\lambda_1(|\xi|)t) \hat{n}_0$  being the remainder terms on right hand side of (3.53). Then

$$|(S_1(t,\xi)\hat{f}_0, v\sqrt{M})|^2 \geq \frac{1}{2|\xi|^2} e^{2\operatorname{Re}\lambda_1(|\xi|)t} \sin^2(\operatorname{Im}\lambda_1(|\xi|)t) |\hat{n}_0|^2 - C|T_5(t,\xi)\hat{f}_0|^2$$

$$\begin{aligned} &\geq \frac{1}{2|\xi|^2} e^{2\operatorname{Re}\lambda_1(|\xi|)t} \sin^2(\operatorname{Im}\lambda_1(|\xi|)t) |\hat{n}_0|^2 \\ &\quad - C e^{-2\beta|\xi|^2 t} (|\hat{n}_0|^2 + |\hat{m}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2). \end{aligned}$$

Similar to (3.78), we get

$$\begin{aligned} \|(S_1(t, \xi) \hat{f}_0, v\sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 &\geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} \frac{1}{|\xi|^2} e^{-2\eta|\xi|^2 t} \sin^2(t + b_1|\xi|^2 t) d\xi \\ &\quad - C \int_{|\xi| \leq r_0} e^{-2\beta|\xi|^2 t} (|\hat{n}_0|^2 + |\xi|^4 t^2 |\hat{n}_0|^2 + |\hat{m}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2) d\xi \\ &\geq C_3(1+t)^{-1/2} - C(1+t)^{-3/2}, \end{aligned} \quad (3.79)$$

which together with (3.73) for  $j = 1, 2, 3$  lead to (3.62) for  $t > 0$  being large enough.

By (3.54) and the fact that  $\lambda_0(|\xi|)$  is real, we have

$$\begin{aligned} |(S_1(t, \xi) \hat{f}_0, \chi_4)|^2 &\geq \frac{1}{2} e^{2\lambda_0(|\xi|)t} |\hat{q}_0|^2 - \frac{2}{3} \left| e^{\operatorname{Re}\lambda_1(|\xi|)t} \cos(\operatorname{Im}\lambda_1(|\xi|)t) - e^{\lambda_0(|\xi|)t} \right|^2 |\hat{n}_0|^2 \\ &\quad - C |\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2. \end{aligned}$$

Then

$$\begin{aligned} \|(S_1(t, \xi) \hat{f}_0, \chi_4)\|_{L^2(\mathbb{R}_\xi^3)}^2 &\geq \frac{1}{2} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2 t} |\hat{q}_0|^2 d\xi - C \int_{|\xi| \leq r_0} e^{-2\beta|\xi|^2 t} (|\hat{n}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2) d\xi \\ &\geq C_3 \left[ \inf_{|\xi| \leq r_0} |\hat{q}_0(\xi)|^2 (1+t)^{-3/2} - d_1 \sup_{|\xi| \leq r_0} |\hat{n}_0(\xi)| (1+t)^{-3/2} \right] - C(1+t)^{-5/2}. \end{aligned}$$

This and (3.73) with  $j = 4$  lead to (3.63) for  $t > 0$  being large enough.

By (3.59), we have

$$\mathbf{P}_1(S_1(t, \xi) \hat{f}_0) = e^{\operatorname{Re}\lambda_1(|\xi|)t} \hat{n}_0 \left[ \sin(\operatorname{Im}\lambda_1(|\xi|)t) L\Psi + \cos(\operatorname{Im}\lambda_1(|\xi|)t) \Psi \right] + |\xi| \sum_{j=-1}^3 e^{t\lambda_j(|\xi|)} \mathbf{P}_1(T_j(\xi) \hat{f}_0),$$

where  $\Psi \in N_0^\perp$  is a non-zero real function given by

$$\Psi = (L - i\mathbf{P}_1)^{-1} (L + i\mathbf{P}_1)^{-1} \mathbf{P}_1(v \cdot \omega)^2 \sqrt{M} \neq 0.$$

Thus, direct computation yields

$$\begin{aligned} \|\mathbf{P}_1(S_1(t, \xi) \hat{f}_0)\|_{L^2(\mathbb{R}_v^3)}^2 &\geq \frac{1}{2} |\hat{n}_0|^2 e^{2\operatorname{Re}\lambda_1(|\xi|)t} \|\sin(\operatorname{Im}\lambda_1(|\xi|)t) L\Psi + \cos(\operatorname{Im}\lambda_1(|\xi|)t) \Psi\|_{L^2(\mathbb{R}_v^3)}^2 \\ &\quad - C |\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 \\ &\geq \frac{1}{4} |\hat{n}_0|^2 e^{-2\eta|\xi|^2 t} \|\sin(t + b_1|\xi|^2 t) L\Psi + \cos(t + b_1|\xi|^2 t) \Psi\|_{L^2(\mathbb{R}_v^3)}^2 \\ &\quad - C e^{-2\beta|\xi|^2 t} (|\xi|^3 t^2 |\hat{n}_0|^2 - C |\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2). \end{aligned}$$

This leads to

$$\begin{aligned} \|\mathbf{P}_1(S_1(t, \xi) \hat{f}_0)\|_{L^2(\mathbb{R}_{\xi, v}^6)}^2 &\geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2 t} \|\sin(t + b_1|\xi|^2 t) L\Psi + \cos(t + b_1|\xi|^2 t) \Psi\|_{L^2(\mathbb{R}_v^3)}^2 d\xi \\ &\quad - C \int_{|\xi| \leq r_0} e^{-2\beta|\xi|^2 t} [(|\xi|^3 t^2 |\hat{n}_0|^2 + |\xi|^2 \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2)] d\xi \\ &\geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2 t} \|\sin(t + b_1|\xi|^2 t) L\Psi + \cos(t + b_1|\xi|^2 t) \Psi\|_{L^2(\mathbb{R}_v^3)}^2 d\xi \\ &\quad - C (\|n_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|f_0\|_{L^{2,1}})(1+t)^{-5/2} \end{aligned}$$

$$=: I_2 - C(\|n_0\|_{L^1(\mathbb{R}_x^3)}^2 + \|f_0\|_{L^{2,1}})(1+t)^{-5/2}, \quad (3.80)$$

where  $C > 0$  and  $\beta > 0$  are two generic positive constants. The term  $I_2$  can be estimated as follows. We obtain for time  $t \geq t_0 =: \frac{L^2}{r_0^2}$  with the constant  $L \geq \sqrt{\frac{4\pi}{b_1}}$  that

$$\begin{aligned} I_2 &= \frac{d_0^2}{4} t^{-3/2} \int_{|\zeta| \leq r_0 \sqrt{t}} e^{-2\eta|\zeta|^2} \|\sin(t + b_1|\zeta|^2)L\Psi + \cos(t + b_1|\zeta|^2)\Psi\|_{L^2(\mathbb{R}_v^3)}^2 d\zeta \\ &\geq \pi d_0^2 (1+t)^{-3/2} \int_0^L r^2 e^{-2\eta r^2} \|\sin(t + b_1 r^2)L\Psi + \cos(t + b_1 r^2)\Psi\|_{L^2(\mathbb{R}_v^3)}^2 dr \\ &=: \pi d_0^2 (1+t)^{-3/2} F_2(t). \end{aligned} \quad (3.81)$$

Since for any  $t \geq t_0$  there exists an integer  $k \geq 0$  so that  $[2k\pi, (2k+2)\pi] \subset [t + \frac{b_1}{4}L^2, t + b_1L^2]$ , we can estimate the lower bound of  $F_2(t)$  as follows

$$\begin{aligned} F_2(t) &\geq \frac{L}{2} e^{-2\eta L^2} \int_{\frac{L}{2}}^L \|\sin(t + b_1 r^2)L\Psi + \cos(t + b_1 r^2)\Psi\|_{L^2(\mathbb{R}_v^3)}^2 r dr \\ &= \frac{L}{4b_1} e^{-2\eta L^2} \int_{t + \frac{b_1}{4}L^2}^{t + b_1 L^2} \|L\Psi \sin y + \Psi \cos y\|_{L^2(\mathbb{R}_v^3)}^2 dy \\ &\geq \frac{L}{4b_1} e^{-2\eta L^2} \int_{2k\pi}^{(2k+2)\pi} \|L\Psi \sin y + \Psi \cos y\|_{L^2(\mathbb{R}_v^3)}^2 dy = \frac{L}{4b_1} e^{-2\eta L^2} \int_0^{2\pi} \|L\Psi \sin y + \Psi \cos y\|_{L^2(\mathbb{R}_v^3)}^2 dy \\ &\geq \frac{L}{4b_1} e^{-2\eta L^2} \int_{\frac{\pi}{2}}^{\pi} \|L\Psi \sin y + \Psi \cos y\|_{L^2(\mathbb{R}_v^3)}^2 dy \\ &= \frac{L}{4b_1} e^{-2\eta L^2} \int_{\frac{\pi}{2}}^{\pi} \left[ \|L\Psi\|_{L^2(\mathbb{R}_v^3)}^2 \sin^2 y + \|\Psi\|_{L^2(\mathbb{R}_v^3)}^2 \cos^2 y + (L\Psi, \Psi) \sin 2y \right] dy \\ &\geq \frac{L}{4b_1} e^{-2\eta L^2} \|\Psi\|_{L^2(\mathbb{R}_v^3)}^2 \int_{\frac{\pi}{2}}^{\pi} \cos^2 y dy = \frac{L\pi}{8b_1} e^{-2\eta L^2} \|\Psi\|_{L^2(\mathbb{R}_v^3)}^2 > 0, \quad t \geq t_0. \end{aligned} \quad (3.82)$$

Substituting (3.82) into (3.81), we obtain

$$I_2 \geq \frac{L d_0^2 \pi^2}{8b_1} e^{-2\eta L^2} \|\Psi\|_{L^2(\mathbb{R}_v^3)}^2 (1+t)^{-3/2},$$

which together with (3.80) and (3.75) imply (3.65) for sufficiently large  $t \geq t_0$ .

By (3.61)–(3.65) and Theorem 3.5, we have (3.66) for  $t > 0$  being sufficiently large by the fact that

$$\begin{aligned} \|e^{tB} f_0\|_{H_P^1} &\geq \|\mathbf{P}_0(e^{tB} f_0)\|_{L_P^2} - \|\mathbf{P}_1(e^{tB} f_0)\|_{L^2(\mathbb{R}_{x,v}^6)} - C \sum_{1 \leq |\alpha| \leq l} \|\partial_x^\alpha e^{tB} f_0\|_{L_P^2} \\ &\geq C_1(1+t)^{-\frac{1}{4}} - C(1+t)^{-\frac{3}{4}} \geq C(1+t)^{-\frac{1}{4}}. \end{aligned}$$

Next, we turn to deal with (3.67)–(3.71) for the case  $(\hat{f}_0, \chi_0) = 0$ . Indeed, by (3.52)–(3.54) we can have

$$(S_1(t, \xi) \hat{f}_0, \sqrt{M}) = -\frac{1}{2} |\xi| \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} j(\hat{m}_0 \cdot \omega) + |\xi|^2 (T_7(t, \xi) \hat{f}_0, \sqrt{M}), \quad (3.83)$$

$$(S_1(t, \xi) \hat{f}_0, v\sqrt{M}) = \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} (\hat{m}_0 \cdot \omega) \omega + \sum_{j=2,3} e^{\lambda_j(|\xi|)t} (\hat{m}_0 \cdot W^j) W^j + |\xi| (T_6(t, \xi) \hat{f}_0, v\sqrt{M}), \quad (3.84)$$

$$(S_1(t, \xi) \hat{f}_0, \chi_4) = e^{\lambda_0(|\xi|)t} \hat{q}_0 + |\xi| (T_6(t, \xi) \hat{f}_0, \chi_4), \quad (3.85)$$

$$\begin{aligned} \mathbf{P}_1(S_1(t, \xi) \hat{f}_0) &= \frac{i}{2} |\xi| \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} (\hat{m}_0 \cdot \omega) (L - j\mathbf{P}_1)^{-1} \mathbf{P}_1(v \cdot \omega)^2 \sqrt{M} + i|\xi| e^{\lambda_0(|\xi|)t} \hat{q}_0 L^{-1} \mathbf{P}_1(v \cdot \omega) \chi_4 \\ &\quad + i|\xi| \sum_{j=2,3} e^{\lambda_j(|\xi|)t} (\hat{m}_0 \cdot W^j) L^{-1} \mathbf{P}_1(v \cdot \omega) (v \cdot W^j) \sqrt{M} + |\xi|^2 \mathbf{P}_1(T_6(t, \xi) \hat{f}_0), \end{aligned} \quad (3.86)$$

where  $T_j(t, \xi)\hat{f}_0$  for  $j = 6, 7$  is the remainder term satisfying  $\|T_j(t, \xi)\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2 \leq Ce^{-2\beta|\xi|^2 t}\|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2$ . Since the vectors  $W^2, W^3$  and  $\omega$  are orthogonal to each other, and the terms  $(L \pm i\mathbf{P}_1)^{-1}\mathbf{P}_1(v \cdot \omega)^2\sqrt{M}$ ,  $L^{-1}\mathbf{P}_1(v \cdot \omega)\chi_4$ ,  $L^{-1}\mathbf{P}_1(v \cdot \omega)(v \cdot W^2)\sqrt{M}$  and  $L^{-1}\mathbf{P}_1(v \cdot \omega)(v \cdot W^3)\sqrt{M}$  are orthogonal. Hence, we get from (3.83)–(3.86) that

$$\begin{aligned} |(S_1(t, \xi)\hat{f}_0, \sqrt{M})|^2 &\geq \frac{1}{2}|\xi|^2 e^{2\operatorname{Re}\lambda_1(|\xi|)t} \sin^2(\operatorname{Im}\lambda_1(|\xi|)t) |(\hat{m}_0 \cdot \omega)|^2 - C|\xi|^4 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2, \\ |(S_1(t, \xi)\hat{f}_0, v\sqrt{M})|^2 &\geq \frac{1}{2}e^{2\operatorname{Re}\lambda_1(|\xi|)t} \cos^2(\operatorname{Im}\lambda_1(|\xi|)t) |(\hat{m}_0 \cdot \omega)|^2 + \frac{1}{2} \sum_{j=2,3} e^{2\operatorname{Re}\lambda_j(|\xi|)t} |(\hat{m}_0 \cdot W^j)|^2 \\ &\quad - C|\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2, \\ |(S_1(t, \xi)\hat{f}_0, \chi_4)|^2 &\geq \frac{1}{2}e^{2\lambda_0(|\xi|)t} |\hat{q}_0|^2 - C|\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2, \\ \|\mathbf{P}_1(S_1(t, \xi)\hat{f}_0)\|_{L^2(\mathbb{R}_v^3)}^2 &\geq \frac{1}{2}\|L^{-1}\mathbf{P}_1(v_1\chi_2)\|_{L^2(\mathbb{R}_v^3)}^2 |\xi|^2 \sum_{j=2,3} e^{2\operatorname{Re}\lambda_j(|\xi|)t} |(\hat{m}_0 \cdot W^j)|^2 \\ &\quad + \frac{1}{2}\|L^{-1}\mathbf{P}_1(v_1\chi_4)\|_{L^2(\mathbb{R}_v^3)}^2 |\xi|^2 e^{2\lambda_0(|\xi|)t} |\hat{q}_0|^2 - C|\xi|^2 e^{-2\beta|\xi|^2 t} \|\hat{f}_0\|_{L^2(\mathbb{R}_v^3)}^2. \end{aligned}$$

This together with the assumptions that  $\inf_{|\xi| \leq r_0} |\hat{m}_0 \cdot \omega| \geq d_0$  and  $\inf_{|\xi| \leq r_0} |\hat{q}_0| \geq d_0$  give

$$\begin{aligned} \|(S_1(t, \xi)\hat{f}_0, \sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 &\geq C_6(1+t)^{-5/2}, \quad \|\mathbf{P}_1(S_1(t, \xi)\hat{f}_0)\|_{L^2(\mathbb{R}_{\xi,v}^6)}^2 \geq C_6(1+t)^{-5/2}, \\ \|(S_1(t, \xi)\hat{f}_0, v\sqrt{M})\|_{L^2(\mathbb{R}_\xi^3)}^2 &\geq C_6(1+t)^{-3/2}, \quad \|(S_1(t, \xi)\hat{f}_0, \chi_4)\|_{L^2(\mathbb{R}_\xi^3)}^2 \geq C_6(1+t)^{-3/2}. \end{aligned}$$

With this, (3.73), (3.74) and (3.75) imply (3.67)–(3.70). The proof is then completed.  $\square$

## 4 The original nonlinear problem

In this section, we prove the long time decay rates of the solution to the Cauchy problem for Vlasov-Poisson-Boltzmann system with the help of the asymptotic behaviors of linearized problem established in Section 3.

### 4.1 Hard sphere case

Denote the weighted function  $w(v)$  by

$$w(v) = (1 + |v|^2)^{1/2}$$

and the Sobolev spaces  $H^N$  and  $H_w^N$  as

$$H^N = \{f \in L^2(\mathbb{R}_{x,v}^6) \mid \|f\|_{H^N} < \infty\}, \quad H_w^N = \{f \in L^2(\mathbb{R}_{x,v}^6) \mid \|f\|_{H_w^N} < \infty\}$$

with the norms

$$\|f\|_{H^N} = \sum_{|\alpha|+|\beta| \leq N} \|\partial_x^\alpha \partial_v^\beta f\|_{L^2(\mathbb{R}_{x,v}^6)}, \quad \|f\|_{H_w^N} = \sum_{|\alpha|+|\beta| \leq N} \|w(v) \partial_x^\alpha \partial_v^\beta f\|_{L^2(\mathbb{R}_{x,v}^6)}.$$

For the hard sphere model, we will prove

**Theorem 4.1.** *Assume that  $f_0 \in H^N \cap L^{2,1}$  with  $N \geq 4$ , and  $\|f_0\|_{H_w^N \cap L^{2,1}} \leq \delta_0$  with  $\delta_0 > 0$  being small enough. Let  $f$  be a solution of the VPB system (1.5). Then, it holds for  $k = 0, 1$  that*

$$\begin{cases} \|\partial_x^k(f(t), \chi_0)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \\ \|\partial_x^k(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad j = 1, 2, 3, \\ \|\partial_x^k(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + \|\partial_x^k \nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \\ \|\mathbf{P}_1 f(t)\|_{H_w^N} + \|\nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))} \leq C\delta_0(1+t)^{-\frac{3}{4}}, \end{cases} \quad (4.1)$$

where  $\chi_j$ ,  $j = 0, 1, 2, 3, 4$ , is defined by (1.11). Moreover, if  $(f_0, \chi_0) = 0$ , then it holds for  $k = 0, 1$  that

$$\begin{cases} \|\partial_x^k(f(t), \chi_0)\|_{L^2(\mathbb{R}_x^3)} + \|\partial_x^k \mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \\ \|\partial_x^k(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 1, 2, 3, \\ \|\partial_x^k(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + \|\partial_x^k \nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \\ \|\mathbf{P}_1 f(t)\|_{H_w^N} + \|\nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))} \leq C\delta_0(1+t)^{-\frac{5}{4}}. \end{cases} \quad (4.2)$$

*Proof.* Let  $f$  be a solution to the IVP problem (1.5) for  $t > 0$ . We can represent this solution in terms of the semigroup  $e^{tB}$  as

$$f(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} G(s) ds, \quad (4.3)$$

where the nonlinear term  $G$  is given by (1.7). For this global solution  $f$ , we define two functionals  $Q_1(t)$  and  $Q_2(t)$  for any  $t > 0$  as

$$\begin{aligned} Q_1(t) = \sup_{0 \leq s \leq t} \sum_{k=0}^1 & \left\{ (1+s)^{\frac{3}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{1}{4}+\frac{k}{2}} \|\partial_x^k(f(s), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \right. \\ & + (1+s)^{\frac{1}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{1}{4}} \|\nabla_x \Phi(s)\|_{L^2(\mathbb{R}_x^3)} \\ & \left. + (1+s)^{\frac{3}{4}} (\|\mathbf{P}_1 f(s)\|_{H_w^N} + \|\nabla_x \mathbf{P}_0 f(s)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))}) \right\}, \end{aligned}$$

and

$$\begin{aligned} Q_2(t) = \sup_{0 \leq s \leq t} \sum_{k=0}^1 & \left\{ (1+s)^{\frac{5}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{3}{4}+\frac{k}{2}} \|\partial_x^k(f(s), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \right. \\ & + (1+s)^{\frac{3}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{3}{4}} \|\nabla_x \Phi(s)\|_{L^2(\mathbb{R}_x^3)} \\ & + (1+s)^{\frac{5}{4}+\frac{k}{2}} \|\partial_x^k \mathbf{P}_1 f(s)\|_{L^2(\mathbb{R}_{x,v}^6)} \\ & \left. + (1+s)^{\frac{5}{4}} (\|\mathbf{P}_1 f(s)\|_{H_w^N} + \|\nabla_x \mathbf{P}_0 f(s)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))}) \right\}. \end{aligned}$$

We claim that it holds under the assumptions of Theorem 4.1 that

$$Q_1(t) \leq C\delta_0, \quad (4.4)$$

and if it is further satisfied  $(f_0, \chi_0) = 0$  that

$$Q_2(t) \leq C\delta_0. \quad (4.5)$$

It is easy to verify that the estimates (4.1) and (4.2) follow from (4.4) from (4.5) respectively.

*First of all, we prove the claim (4.4) as follows.* To begin with, let us deal with the time-decay rate of the macroscopic density, momentum and energy, which in terms of (4.3) satisfy the following equations

$$(f(t), \chi_j) = (e^{tB} f_0, \chi_j) + \int_0^t (e^{(t-s)B} G(s), \chi_j) ds, \quad j = 0, 1, 2, 3, 4. \quad (4.6)$$

In the case of  $(f_0, \sqrt{M}) = 0$ , we can obtain by (3.83)–(3.86) that

$$\|\partial_x^\alpha(e^{tB} f_0, \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{1}{2}-\frac{|\alpha|}{2}} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|(f_0, v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} + \|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)}), \quad (4.7)$$

$$\|\partial_x^\alpha(e^{tB} f_0, \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x f_0\|_{L^{2,1}}), \quad (4.8)$$

$$\|\partial_x^\alpha(e^{tB} f_0, v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|(f_0, v\sqrt{M})\|_{L^1(\mathbb{R}_x^3)} + \|\nabla_x f_0\|_{L^{2,1}}), \quad (4.9)$$

$$\|\partial_x^\alpha(e^{tB} f_0, \chi_4)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|(f_0, \chi_4)\|_{L^1(\mathbb{R}_x^3)} + \|\nabla_x f_0\|_{L^{2,1}}), \quad (4.10)$$

$$\|\mathbf{P}_1(\partial_x^\alpha e^{tB} f_0)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}} (\|\partial_x^\alpha f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x f_0\|_{L^{2,1}}), \quad (4.11)$$

for  $|\alpha| \geq 0$ .

By (4.7)–(4.11) for  $|\alpha| = 0, 1$ , we are able to establish the a-priori estimates of the nonlinear terms in the right hand side of (4.3) as follows. Since the term  $\Gamma(f, g)$  satisfies (cf.[4])

$$\|\Gamma(f, g)\|_{L^2(\mathbb{R}_v^3)} \leq C(\|f\|_{L^2(\mathbb{R}_v^3)}\|\nu g\|_{L^2(\mathbb{R}_v^3)} + \|\nu f\|_{L^2(\mathbb{R}_v^3)}\|g\|_{L^2(\mathbb{R}_v^3)}), \quad (4.12)$$

$$\|\Gamma(f, g)\|_{L^{2,1}} \leq C(\|f\|_{L^2(\mathbb{R}_{x,v}^6)}\|\nu g\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nu f\|_{L^2(\mathbb{R}_{x,v}^6)}\|g\|_{L^2(\mathbb{R}_{x,v}^6)}), \quad (4.13)$$

we can estimate the nonlinear term  $G(s)$  given by (1.7) for  $0 \leq s \leq t$  in terms of  $Q_1(t)$  as

$$\begin{aligned} \|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} &\leq C\{\|wf\|_{L^{2,3}}\|f\|_{L^{2,6}} + \|\nabla_x \Phi\|_{L^3(\mathbb{R}_x^3)}(\|wf\|_{L^{2,6}} + \|\nabla_v f\|_{L^{2,6}})\} \\ &\leq C(1+s)^{-1}Q_1(t)^2, \end{aligned} \quad (4.14)$$

$$\begin{aligned} \|G(s)\|_{L^{2,1}} &\leq C\{\|f\|_{L^2(\mathbb{R}_{x,v}^6)}\|wf\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x \Phi\|_{L^2(\mathbb{R}_x^3)}(\|wf\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_v f\|_{L^2(\mathbb{R}_{x,v}^6)})\} \\ &\leq C(1+s)^{-\frac{1}{2}}Q_1(t)^2, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \|\nabla_x G(s)\|_{L^{2,1}} &\leq C\{\|\nabla_x f\|_{L^2(\mathbb{R}_{x,v}^6)}\|wf\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f\|_{L^2(\mathbb{R}_{x,v}^6)}\|w\nabla_x f\|_{L^2(\mathbb{R}_{x,v}^6)} \\ &\quad + \|\partial_x^2 \Phi\|_{L^2(\mathbb{R}_x^3)}(\|wf\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_v f\|_{L^2(\mathbb{R}_{x,v}^6)}) \\ &\quad + \|\nabla_x \Phi\|_{L^2(\mathbb{R}_x^3)}(\|w\nabla_x f\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_v \nabla_x f\|_{L^2(\mathbb{R}_{x,v}^6)})\} \\ &\leq C(1+s)^{-1}Q_1(t)^2. \end{aligned} \quad (4.16)$$

Noting further that it holds  $(G, \chi_0) = 0$ , we obtain by (3.35), (3.40), (4.8), and (4.14)–(4.16) the long time decay rate of the macroscopic density  $(f(t), \chi_0)$  as

$$\begin{aligned} \|(f(t), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\frac{3}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\ &\quad + C \int_0^{t/2} (1+t-s)^{-\frac{5}{4}}(\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}})ds \\ &\quad + C \int_{t/2}^t (1+t-s)^{-\frac{3}{4}}(\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\ &\leq C\delta_0(1+t)^{-\frac{3}{4}} + C \int_0^{t/2} (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{1}{2}}Q_1(t)^2ds \\ &\quad + C \int_{t/2}^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-1}Q_1(t)^2ds \\ &\leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}Q_1(t)^2. \end{aligned} \quad (4.17)$$

Meanwhile, we have by (3.40) and (4.7) the long time decay rate of the spatial derivative  $(\nabla_x f(t), \chi_0)$  of the macroscopic density as

$$\begin{aligned} \|(\nabla_x f(t), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\frac{5}{4}}(\|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\ &\quad + C \int_0^{t/2} (1+t-s)^{-\frac{7}{4}}(\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}})ds \\ &\quad + C \int_{t/2}^t (1+t-s)^{-1}(\|(G(s), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} + \|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)})ds \\ &\leq C\delta_0(1+t)^{-\frac{5}{4}} + C \int_0^{t/2} (1+t-s)^{-\frac{7}{4}}(1+s)^{-\frac{1}{2}}Q_1(t)^2ds \\ &\quad + C \int_{t/2}^t (1+t-s)^{-1}(1+s)^{-\frac{3}{2}}Q_1(t)^2ds \\ &\leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}Q_1(t)^2, \end{aligned} \quad (4.18)$$

where we have used the following estimates

$$\|(G(s), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} + \|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C(1+s)^{-\frac{3}{2}}Q_1(t)^2, \quad 0 \leq s \leq t.$$

Similarly, in terms of (3.36) and (4.9) we can establish the a-priori estimates on the long time decay rates of the macroscopic momentum  $(f(t), v\sqrt{M})$  and its spatial derivative as

$$\begin{aligned}
& \|(f(t), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \\
& \leq C(1+t)^{-\frac{1}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& \quad + C \int_0^t (1+t-s)^{-\frac{3}{4}}(\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|(G(s), v\sqrt{M})\|_{L^1(\mathbb{R}_x^3)} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\
& \leq C\delta_0(1+t)^{-\frac{1}{4}} + C \int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-1}Q_1(t)^2 ds \\
& \leq C\delta_0(1+t)^{-\frac{1}{4}} + C(1+t)^{-\frac{3}{4}+\varepsilon}Q_1(t)^2,
\end{aligned} \tag{4.19}$$

where  $\varepsilon > 0$  is a small but fixed constant. Here, we have used the fact that  $\|(G(s), v\sqrt{M})\|_{L^1(\mathbb{R}_x^3)} \leq C(1+s)^{-1}Q_1(t)^2$ , and

$$\begin{aligned}
& \|\nabla_x(f(t), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \\
& \leq C(1+t)^{-\frac{3}{4}}(\|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\partial_x^\alpha f_0\|_{L^{2,1}}) \\
& \quad + C \int_0^t (1+t-s)^{-\frac{5}{4}}(\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|(G(s), v\sqrt{M})\|_{L^1(\mathbb{R}_x^3)} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\
& \leq C\delta_0(1+t)^{-\frac{3}{4}} + C \int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-1}Q_1(t)^2 ds \\
& \leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-1}Q_1(t)^2.
\end{aligned} \tag{4.20}$$

In terms of (3.37), (4.10), and

$$\|(G(s), \chi_4)\|_{L^1(\mathbb{R}_x^3)} \leq C(1+s)^{-\frac{1}{2}}Q_1(t)^2, \quad \|\nabla_x(G(s), \chi_4)\|_{L^1(\mathbb{R}_x^3)} \leq C(1+s)^{-1}Q_1(t)^2,$$

we can estimate the macroscopic energy  $(f(t), \chi_4)$  and its spatial derivative as

$$\begin{aligned}
& \|(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} \\
& \leq C(1+t)^{-\frac{3}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) + C \int_0^t (1+t-s)^{-\frac{3}{4}}\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)}ds \\
& \quad + C \int_0^t (1+t-s)^{-\frac{3}{4}}(\|(G(s), \chi_4)\|_{L^1(\mathbb{R}_x^3)} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\
& \leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{1}{4}}Q_1(t)^2,
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
& \|\nabla_x(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} \\
& \leq C(1+t)^{-\frac{5}{4}}(\|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& \quad + C \int_0^{t/2} (1+t-s)^{-\frac{5}{4}}(\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|(G(s), \chi_4)\|_{L^1(\mathbb{R}_x^3)} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\
& \quad + C \int_{t/2}^t (1+t-s)^{-\frac{3}{4}}(\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x(G(s), \chi_4)\|_{L^1(\mathbb{R}_x^3)} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\
& \leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{3}{4}}Q_1(t)^2.
\end{aligned} \tag{4.22}$$

Moreover, the electricity potential  $\nabla_x \Phi$  is bounded by

$$\begin{aligned}
\|\nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} & \leq C(1+t)^{-\frac{1}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& \quad + C \int_0^t (1+t-s)^{-\frac{3}{4}}(\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}})ds
\end{aligned}$$



$$\leq C\delta_0(1+t)^{-\frac{1}{4}} + C(1+t)^{-\frac{1}{4}}Q_1(t)^2. \quad (4.23)$$

In addition, the microscopic part  $\mathbf{P}_1 f(t)$  can be estimated by (3.39), (3.44) and (4.11) as follows

$$\begin{aligned} \|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} &\leq C(1+t)^{-\frac{3}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\ &\quad + \int_0^{t/2} (1+t-s)^{-\frac{5}{4}}(\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}})ds \\ &\quad + \int_{t/2}^t (1+t-s)^{-\frac{3}{4}}(\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x G(s)\|_{L^{2,1}})ds \\ &\leq C\delta_0(1+t)^{-\frac{3}{4}} + C \int_0^{t/2} (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{1}{2}}Q_1(t)^2 ds \\ &\quad + \int_{t/2}^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-1}Q_1(t)^2 ds \\ &\leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}Q_1(t)^2. \end{aligned} \quad (4.24)$$

With the help of the apriori estimates (4.17)–(4.23), we are able to verify the claim (4.4). Indeed, similar to that of Lemma 4.6 in [2], we claim that there are two functionals  $H(f)$  and  $D(f)$  related to the global solution  $f$ :

$$\begin{aligned} H(f) &\sim \sum_{|\alpha|+|\beta|\leq N} \|w\partial_x^\alpha \partial_v^\beta \mathbf{P}_1 f\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \sum_{|\alpha|\leq N-1} (\|\partial_x^\alpha \nabla_x \mathbf{P}_0 f\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \|\partial_x^\alpha P_d f\|_{L^2(\mathbb{R}_{x,v}^6)}^2), \\ D(f) &\sim \sum_{|\alpha|+|\beta|\leq N} \|w^{\frac{3}{2}}\partial_x^\alpha \partial_v^\beta \mathbf{P}_1 f\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \sum_{|\alpha|\leq N-1} (\|\partial_x^\alpha \nabla_x \mathbf{P}_0 f\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \|\partial_x^\alpha P_d f\|_{L^2(\mathbb{R}_{x,v}^6)}^2), \end{aligned} \quad (4.25)$$

such that

$$\frac{d}{dt}H(f(t)) + \mu D(f(t)) \leq C\|\nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2. \quad (4.26)$$

This together with (4.18), (4.20), and (4.22) leads to

$$\begin{aligned} H(f(t)) &\leq e^{-\mu t}H(f_0) + \int_0^t e^{-\mu(t-s)}\|\nabla_x \mathbf{P}_0 f(s)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 ds \\ &\leq C\delta_0^2 e^{-\mu t} + \int_0^t e^{-\mu(t-s)}(1+s)^{-\frac{3}{2}}(\delta_0 + Q_1(t)^2)^2 ds \\ &\leq C(1+t)^{-\frac{3}{2}}(\delta_0 + Q_1(t)^2)^2. \end{aligned} \quad (4.27)$$

Making summation of (4.17)–(4.23) and (4.27), we have

$$Q_1(t) \leq C\delta_0 + CQ_1(t)^2,$$

from which the claim (4.4) can be verified provided that  $\delta_0 > 0$  is small enough.

Next, we turn to prove the claim (4.5) for the case  $(f_0, \chi_0) = 0$  as follows. Indeed, if it holds  $(f_0, \chi_0) = 0$ , the time decay rates of the macroscopic density, momentum and energy on the right hand side of (4.6) can be estimated by using (3.40)–(3.43) as follows. In fact, the macroscopic density and its spatial derivative can be estimated by

$$\begin{aligned} \|(f(t), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\frac{5}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\ &\quad + C \int_0^t (1+t-s)^{-\frac{5}{4}}(\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}})ds \\ &\leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}Q_2(t)^2, \\ \|\nabla_x(f(t), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\frac{7}{4}}(\|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \end{aligned} \quad (4.28)$$

$$\begin{aligned}
& + C \int_0^{t/2} (1+t-s)^{-\frac{7}{4}} (\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds \\
& + C \int_{t/2}^t (1+t-s)^{-\frac{5}{4}} (\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x G(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{7}{4}} + C(1+t)^{-\frac{7}{4}}Q_2(t)^2,
\end{aligned} \tag{4.29}$$

where we have used the fact that

$$\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}} \leq C(1+s)^{-\frac{3}{2}}Q_2(t)^2, \quad \|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x G(s)\|_{L^{2,1}} \leq C(1+s)^{-2}Q_2(t)^2,$$

because of (4.14), (4.15), and (4.16). In terms of (3.41), the macroscopic momentum and its spatial derivative can be estimated as

$$\begin{aligned}
\|(f(t), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} & \leq C(1+t)^{-\frac{3}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& + C \int_0^t (1+t-s)^{-\frac{3}{4}} (\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}Q_2(t)^2,
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\|\nabla_x(f(t), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} & \leq C(1+t)^{-\frac{5}{4}}(\|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& + C \int_0^t (1+t-s)^{-\frac{5}{4}} (\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}Q_2(t)^2.
\end{aligned} \tag{4.31}$$

Furthermore, the macroscopic energy and its spatial derivative can be bounded in terms of (3.42) as

$$\begin{aligned}
\|(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} & \leq C(1+t)^{-\frac{3}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& + C \int_0^t (1+t-s)^{-\frac{3}{4}} (\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}Q_2(t)^2,
\end{aligned} \tag{4.32}$$

$$\begin{aligned}
\|\nabla_x(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} & \leq C(1+t)^{-\frac{5}{4}}(\|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& + C \int_0^t (1+t-s)^{-\frac{5}{4}} (\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}Q_2(t)^2,
\end{aligned} \tag{4.33}$$

and the electric field can be controlled by (3.43) as

$$\begin{aligned}
\|\nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} & \leq C(1+t)^{-\frac{3}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& + C \int_0^t (1+t-s)^{-\frac{3}{4}} (\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{1}{4}} + C(1+t)^{-\frac{1}{4}}Q_2(t)^2.
\end{aligned} \tag{4.34}$$

In addition, the microscopic part of  $f$  can be controlled by (3.44) and (4.11) as

$$\begin{aligned}
\|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} & \leq C(1+t)^{-\frac{5}{4}}(\|f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& + \int_0^t (1+t-s)^{-\frac{5}{4}} (\|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}Q_1(t)^2, \\
\|\nabla_x \mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} & \leq C(1+t)^{-\frac{7}{4}}(\|\nabla_x f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|f_0\|_{L^{2,1}}) \\
& + \int_0^{t/2} (1+t-s)^{-\frac{7}{4}} (\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|G(s)\|_{L^{2,1}}) ds
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
& + \int_{t/2}^t (1+t-s)^{-\frac{5}{4}} (\|\nabla_x G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x G\|_{L^{2,1}}) ds \\
& \leq C\delta_0(1+t)^{-\frac{7}{4}} + C(1+t)^{-\frac{7}{4}} Q_1(t)^2.
\end{aligned} \tag{4.36}$$

Therefore, with the help of (4.28)–(4.33) we can obtain by (4.26) that

$$\begin{aligned}
H(f(t)) & \leq e^{-\mu t} H(f_0) + \int_0^t e^{-\mu(t-s)} \|\nabla_x \mathbf{P}_0 f(s)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 ds \\
& \leq C\delta_0^2 e^{-\mu t} + \int_0^t e^{-\mu(t-s)} (1+s)^{-\frac{5}{2}} (\delta_0 + Q_2(t)^2)^2 ds \\
& \leq C(1+t)^{-\frac{5}{2}} (\delta_0 + Q_2(t)^2)^2.
\end{aligned}$$

This together with (4.25) yields

$$Q_2(t) \leq C\delta_0 + CQ_2(t)^2,$$

which implies the claim (4.5) provided that  $(f_0, \chi_0) = 0$  and  $\delta_0 > 0$  is small enough.  $\square$

Finally, we can establish the optimal time decay rates of the global solution in the following sense.

**Theorem 4.2.** *Assume that  $f_0 \in H^N \cap L^{2,1}$  for  $N \geq 4$  satisfying  $\|f_0\|_{H_w^N \cap L^{2,1}} \leq \delta_0$  with  $\delta_0 > 0$  being small enough, and that there exist two positive constants  $d_0 > 0$  and  $d_1 > 0$  so that  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| \geq d_0$  and  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)|$ . Then, for time  $t > 0$  large enough it holds for the global solution  $f$  to the IVP problem (1.5) that*

$$C_1 \delta_0 (1+t)^{-\frac{3}{4}} \leq \|(f(t), \chi_0)\|_{L^2(\mathbb{R}_x^3)} \leq C_2 \delta_0 (1+t)^{-\frac{3}{4}}, \tag{4.37}$$

$$C_1 \delta_0 (1+t)^{-\frac{1}{4}} \leq \|(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C_2 \delta_0 (1+t)^{-\frac{1}{4}}, \tag{4.38}$$

$$C_1 \delta_0 (1+t)^{-\frac{1}{4}} \leq \|\nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C_2 \delta_0 (1+t)^{-\frac{1}{4}}, \tag{4.39}$$

$$C_1 \delta_0 (1+t)^{-\frac{3}{4}} \leq \|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C_2 \delta_0 (1+t)^{-\frac{3}{4}}, \tag{4.40}$$

$$C_1 \delta_0 (1+t)^{-\frac{1}{4}} \leq \|f(t)\|_{H_w^N} \leq C_2 \delta_0 (1+t)^{-\frac{1}{4}}, \tag{4.41}$$

with  $j = 1, 2, 3$  and  $C_2 \geq C_1 > 0$  being two constants.

If in addition  $(f_0, \chi_0) = 0$  is assumed, and  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, (v \cdot \omega) \sqrt{M})| \geq d_0$  with  $\omega = \frac{\xi}{|\xi|} \in \mathbb{S}^2$ , and  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_0$  for some constant  $d_0 > 0$ . Then, it holds

$$C_1 \delta_0 (1+t)^{-\frac{5}{4}} \leq \|(f(t), \chi_0)\|_{L^2(\mathbb{R}_x^3)} \leq C_2 \delta_0 (1+t)^{-\frac{5}{4}}, \tag{4.42}$$

$$C_1 \delta_0 (1+t)^{-\frac{3}{4}} \leq \|(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C_2 \delta_0 (1+t)^{-\frac{3}{4}}, \tag{4.43}$$

$$C_1 \delta_0 (1+t)^{-\frac{3}{4}} \leq \|(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} \leq C_2 \delta_0 (1+t)^{-\frac{3}{4}}, \tag{4.44}$$

$$C_1 \delta_0 (1+t)^{-\frac{3}{4}} \leq \|\nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C_2 \delta_0 (1+t)^{-\frac{3}{4}}, \tag{4.45}$$

$$C_1 \delta_0 (1+t)^{-\frac{5}{4}} \leq \|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C_2 \delta_0 (1+t)^{-\frac{5}{4}}, \tag{4.46}$$

$$C_1 \delta_0 (1+t)^{-\frac{3}{4}} \leq \|f(t)\|_{H_w^N} \leq C_2 \delta_0 (1+t)^{-\frac{3}{4}}. \tag{4.47}$$

*Proof.* By (4.3), Theorem 3.6 and Theorem 4.1, we can establish the lower bounds of the time decay rates of macroscopic density, momentum and energy of the global solution  $f$  and its microscopic part for  $t > 0$  large enough that

$$\begin{aligned}
\|(f(t), \chi_0)\|_{L^2(\mathbb{R}_x^3)} & \geq \|(e^{tB} f_0, \chi_0)\|_{L^2(\mathbb{R}_x^3)} - \int_0^t \|(e^{(t-s)B} G(s), \chi_0)\|_{L^2(\mathbb{R}_x^3)} ds \\
& \geq C_1 \delta_0 (1+t)^{-3/4} - C_2 \delta_0^2 (1+t)^{-3/4}, \\
\|(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} & \geq \|(e^{tB} f_0, \chi_j)\|_{L^2(\mathbb{R}_x^3)} - \int_0^t \|(e^{(t-s)B} G(s), \chi_j)\|_{L^2(\mathbb{R}_x^3)} ds
\end{aligned}$$

$$\begin{aligned}
&\geq C_1\delta_0(1+t)^{-1/4} - C_2\delta_0^2(1+t)^{-1/4}, \quad j = 1, 2, 3, \\
\|(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} &\geq \|(e^{tB}f_0, \chi_4)\|_{L^2(\mathbb{R}_x^3)} - \int_0^t \|(e^{(t-s)B}G(s), \chi_4)\|_{L^2(\mathbb{R}_x^3)} ds \\
&\geq C_1\delta_0(1+t)^{-3/4} - C_2\delta_0^2(1+t)^{-1/4}, \\
\|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} &\geq \|\mathbf{P}_1(e^{tB}f_0)\|_{L^2(\mathbb{R}_{x,v}^6)} - \int_0^t \|\mathbf{P}_1(e^{(t-s)B}G(s))\|_{L^2(\mathbb{R}_{x,v}^6)} ds \\
&\geq C_1\delta_0(1+t)^{-3/4} - C_2\delta_0^2(1+t)^{-3/4},
\end{aligned} \tag{4.48}$$

from which and Theorem 4.1, we can obtain

$$\begin{aligned}
\|f(t)\|_{H_w^N} &\geq \|\mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} - \|w\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} - \sum_{1 \leq |\alpha| \leq N} \|w\partial_x^\alpha f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \\
&\geq C_1\delta_0(1+t)^{-1/4} - 3C_2\delta_0^2(1+t)^{-1/4} - C_3\delta_0(1+t)^{-3/4}.
\end{aligned}$$

This gives rise to (4.37)–(4.41) for sufficiently large  $t > 0$  and small enough  $\delta_0 > 0$ . (4.43)–(4.47) can be proved similarly so that we omit the detail for brevity.  $\square$

**Remark 4.3.** Let us give an example of the initial function  $f_0$  which satisfies the assumptions of Theorem 4.2. For two positive constants  $d_0$  and  $d_1$ , we define  $f_0(x, v)$  in terms of the orthonormal basis  $\chi_j$ ,  $j = 0, 1, 2, 3, 4$ , as

$$f_0(x, v) = d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|x|^2}{2}} \chi_0(v) + d_1 d_0 e^{r_0^2} e^{-\frac{|x|^2}{2}} \chi_4(v).$$

We can verify that  $f_0$  satisfies the assumptions in the first part of Theorem 4.2 provided that  $d_0 > 0$  is small enough because

$$\begin{aligned}
(\hat{f}_0, \chi_0) &= d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|\xi|^2}{2}}, \quad (\hat{f}_0, \chi_4) = d_1 d_0 e^{r_0^2} e^{-\frac{|\xi|^2}{2}}, \\
\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| &= d_0, \quad \sup_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| = d_0 e^{\frac{r_0^2}{2}}, \quad \inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| = d_0 d_1 e^{\frac{r_0^2}{2}}, \\
\|f_0\|_{H_w^N} &\leq C d_0 e^{\frac{r_0^2}{2}} (1 + d_1 e^{\frac{r_0^2}{2}}) \leq \delta_0.
\end{aligned}$$

In addition, the additional assumption  $(f_0, \chi_0) = 0$  in the second part of Theorem 4.2 is satisfied by

$$f_0(x, v) = d_0 e^{\frac{r_0^2}{2}} (m \cdot v) \sqrt{M} + d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|x|^2}{2}} \chi_4(v),$$

with  $m(x) = \int_{\mathbb{R}^3} \frac{\xi}{|\xi|} e^{-\frac{|\xi|^2}{2}} e^{ix \cdot \xi} d\xi$ . And then for  $d_0 > 0$  being small enough, we have

$$\begin{aligned}
\hat{f}_0 &= d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|\xi|^2}{2}} (v \cdot \omega) \sqrt{M} + d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|\xi|^2}{2}} \chi_4, \\
\inf_{|\xi| \leq r_0} |(\hat{f}_0, (v \cdot \omega) \sqrt{M})| &= d_0, \quad \inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| = d_0, \quad \|f_0\|_{H_w^N} \leq C d_0 e^{\frac{r_0^2}{2}} \leq \delta_0.
\end{aligned}$$

## 4.2 Hard potential case

For hard potential case, we can use a mixed time-velocity weight function introduced in [6] defined by

$$w_l(t, v) = (1 + |v|^2)^{\frac{l}{2}} e^{\frac{a|v|}{(1+t)^b}},$$

where  $l \in \mathbb{R}$ ,  $a > 0$  and  $b > 0$ , and the energy norms

$$\|f(t)\|_{N,l} = \sum_{|\alpha|+|\beta| \leq N} \|w_l(t, v) \partial_x^\alpha \partial_v^\beta f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}, \quad \|f_0\|_{N,l} = \sum_{|\alpha|+|\beta| \leq N} \|w_l(0, v) \partial_x^\alpha \partial_v^\beta f_0\|_{L^2(\mathbb{R}_{x,v}^6)} \tag{4.49}$$

to prove

**Theorem 4.4.** Let  $N \geq 4$ ,  $l \geq 1$ ,  $a > 0$  and  $0 < b \leq 1/4$ . Assume that  $\|f_0\|_{N,l} + \|f_0\|_{L^{2,1}} \leq \delta_0$  with  $\delta_0 > 0$  small. Let  $f$  be a solution of the VPB system (1.5). Then, it holds for  $k = 0, 1$  that

$$\begin{cases} \|\partial_x^k(f(t), \chi_0)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \\ \|\partial_x^k(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad j = 1, 2, 3, \\ \|\partial_x^k(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + \|\partial_x^k \nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \\ \|\mathbf{P}_1 f(t)\|_{N,l} + \|\nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))} \leq C\delta_0(1+t)^{-\frac{3}{4}}. \end{cases} \quad (4.50)$$

Moreover, if  $(f_0, \chi_0) = 0$ , then it holds for  $k = 0, 1$  that

$$\begin{cases} \|\partial_x^k(f(t), \chi_0)\|_{L^2(\mathbb{R}_x^3)} + \|\partial_x^k \mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \\ \|\partial_x^k(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 1, 2, 3, \\ \|\partial_x^k(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + \|\partial_x^k \nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \\ \|\mathbf{P}_1 f(t)\|_{N,l} + \|\nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))} \leq C\delta_0(1+t)^{-\frac{5}{4}}. \end{cases} \quad (4.51)$$

*Proof.* For the global solution  $f$  to the IVP problem (1.5), we define two functionals  $Q_3(t)$  and  $Q_4(t)$  for any  $t > 0$  by

$$\begin{aligned} Q_3(t) = \sup_{0 \leq s \leq t} \sum_{k=0}^1 & \left\{ (1+s)^{\frac{3}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{1}{4}+\frac{k}{2}} \|\partial_x^k(f(s), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \right. \\ & + (1+s)^{\frac{1}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{1}{4}} \|\nabla_x \Phi(s)\|_{L^2(\mathbb{R}_x^3)} \\ & \left. + (1+s)^{\frac{3}{4}} (\|\mathbf{P}_1 f(s)\|_{N,l} + \|\nabla_x \mathbf{P}_0 f(s)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))}) \right\}, \end{aligned}$$

and

$$\begin{aligned} Q_4(t) = \sup_{0 \leq s \leq t} \sum_{k=0}^1 & \left\{ (1+s)^{\frac{5}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{3}{4}+\frac{k}{2}} \|\partial_x^k(f(s), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \right. \\ & + (1+s)^{\frac{3}{4}+\frac{k}{2}} \|\partial_x^k(f(s), \chi_4)\|_{L^2(\mathbb{R}_x^3)} + (1+s)^{\frac{3}{4}} \|\nabla_x \Phi(s)\|_{L^2(\mathbb{R}_x^3)} \\ & + (1+s)^{\frac{5}{4}+\frac{k}{2}} \|\partial_x^k \mathbf{P}_1 f(s)\|_{L^2(\mathbb{R}_{x,v}^6)} \\ & \left. + (1+s)^{\frac{5}{4}} (\|\mathbf{P}_1 f(s)\|_{N,l} + \|\nabla_x \mathbf{P}_0 f(s)\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))}) \right\}. \end{aligned}$$

We claim that it holds under the assumptions of Theorem 4.4 that

$$Q_3(t) \leq C\delta_0, \quad (4.52)$$

and if it is further satisfied  $(f_0, \chi_0) = 0$  that

$$Q_4(t) \leq C\delta_0. \quad (4.53)$$

It is easy to verify that the estimates (4.50) and (4.51) follow from (4.52) from (4.53) respectively.

Since  $\nu(v) \leq w_l(t, v)$  for all  $l \geq 1$  and  $(t, v) \in \mathbb{R}^+ \times \mathbb{R}^3$ , it follows from (4.12) and (4.13) that

$$\begin{aligned} \|\Gamma(f, g)\|_{L^2(\mathbb{R}_v^3)} & \leq C(\|f\|_{L^2(\mathbb{R}_v^3)} \|w_l g\|_{L^2(\mathbb{R}_v^3)} + \|w_l f\|_{L^2(\mathbb{R}_v^3)} \|g\|_{L^2(\mathbb{R}_v^3)}), \\ \|\Gamma(f, g)\|_{L^{2,1}} & \leq C(\|f\|_{L^2(\mathbb{R}_{x,v}^6)} \|w_l g\|_{L^2(\mathbb{R}_{x,v}^6)} + \|w_l f\|_{L^2(\mathbb{R}_{x,v}^6)} \|g\|_{L^2(\mathbb{R}_{x,v}^6)}). \end{aligned}$$

Then, we can obtain by using the similar arguments for (4.14)–(4.16) to obtain

$$\begin{aligned} \|G(s)\|_{L^2(\mathbb{R}_{x,v}^6)} & \leq C(1+s)^{-1} Q_3(t)^2, \\ \|G(s)\|_{L^{2,1}} & \leq C(1+s)^{-\frac{1}{2}} Q_3(t)^2, \\ \|\nabla_x G(s)\|_{L^{2,1}} & \leq C(1+s)^{-1} Q_3(t)^2. \end{aligned}$$

Similar to the proof of Theorem 4.1, we have

$$\|\partial_x^\alpha(f(t), \sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}} + C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}}Q_3(t)^2, \quad (4.54)$$

$$\|\partial_x^\alpha(f(t), v\sqrt{M})\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{2}} + C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{2}}Q_3(t)^2, \quad (4.55)$$

$$\|\partial_x^\alpha(f(t), \chi_4)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}} + C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{2}}Q_3(t)^2, \quad (4.56)$$

for  $|\alpha| = 0, 1$ , and

$$\|\nabla_x \Phi(t)\|_{L^2(\mathbb{R}_x^3)} \leq C\delta_0(1+t)^{-\frac{1}{4}} + C(1+t)^{-\frac{1}{4}}Q_3(t)^2, \quad (4.57)$$

$$\|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^3)} \leq C\delta_0(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}Q_3(t)^2. \quad (4.58)$$

By Lemma 4.4 in [6], there are two functionals  $H_{N,l}(f)$  and  $D_{N,l}(f)$  with

$$H_{N,l}(f(t)) \sim \sum_{|\alpha|+|\beta|\leq N} \|w_l(t, v)\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \sum_{|\alpha|\leq N-1} \|\partial_x^\alpha \nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \|P_d f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2,$$

$$D_{N,l}(f(t)) \sim \sum_{|\alpha|+|\beta|\leq N} \|\nu^{1/2} w_l(t, v)\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \sum_{|\alpha|\leq N-1} \|\partial_x^\alpha \nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \|P_d f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2,$$

such that

$$\frac{d}{dt} H_{N,l}(f(t)) + \kappa D_{N,l}(f(t)) \leq C \|\nabla_x \mathbf{P}_0 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2. \quad (4.59)$$

This together with (4.54), (4.55) and (4.56) leads to

$$\begin{aligned} H_{N,l}(f(t)) &\leq e^{-\kappa t} H_{N,l}(f_0) + \int_0^t e^{-\kappa(t-s)} \|\nabla_x \mathbf{P}_0 f(s)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 ds \\ &\leq e^{-\kappa t} H_{N,l}(f_0) + \int_0^t e^{-\kappa(t-s)} (1+s)^{-3/2} (\delta_0 + Q_3(t)^2)^2 ds \\ &\leq C(1+t)^{-3/2} (\delta_0 + Q_3(t)^2)^2. \end{aligned} \quad (4.60)$$

Summing up (4.54)–(4.58) and (4.60), we have

$$Q_3(t) \leq C\delta_0 + CQ_3(t)^2,$$

from which the claim (4.52) can be verified provided that  $\delta_0 > 0$  is small enough.

Similarly, we can prove (4.51) in the case of  $P_d f_0 = 0$ . The detail is omitted for brevity.  $\square$

Note that the same statements on the optimal decay rates given Theorem 4.2 for hard sphere model hold also for the hard potential case.

## 5 Further discussions

### 5.1 Decay rate of the temperature

In this subsection, we make a comparison of the Vlasov-Poisson-Boltzmann (1.5) system with the classical compressible Navier-Stokes-Poisson equations about the time-asymptotic behavior of the macroscopic density, momentum and the temperature. In order to apply the macro-micro decomposition (1.14) to the system (1.5) to deduce the equations for the macroscopic quantities  $(n, m, q)$ , we take the inner products of  $\chi_j$  ( $j = -1, \dots, 3$ ) and  $(1.5)_1$  respectively to obtain the system of compressible Euler-Poisson type (EP) as

$$\begin{cases} \partial_t n + \operatorname{div}_x m = 0, \\ \partial_t m + \nabla_x n + \sqrt{\frac{2}{3}} \nabla_x q - \nabla_x \Phi = n \nabla_x \Phi - \int_{\mathbb{R}^3} v \cdot \nabla_x (\mathbf{P}_1 f) v \sqrt{M} dv, \\ \partial_t q + \sqrt{\frac{2}{3}} \operatorname{div}_x m = \sqrt{\frac{2}{3}} \nabla_x \Phi \cdot m - \int_{\mathbb{R}^3} v \cdot \nabla_x (\mathbf{P}_1 f) \chi_4 dv. \end{cases} \quad (5.1)$$

Taking the microscopic projection  $\mathbf{P}_1$  to (1.5), we have

$$\partial_t(\mathbf{P}_1 f) + \mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_1 f) - \mathbf{P}_1 G = L(\mathbf{P}_1 f) - \mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_0 f), \quad (5.2)$$

where the nonlinear term  $G$  is denoted by (1.7), and express the microscopic part  $\mathbf{P}_1 f$  as

$$\mathbf{P}_1 f = L^{-1}[\partial_t(\mathbf{P}_1 f) + \mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_1 f) - \mathbf{P}_1 G] + L^{-1}\mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_0 f).$$

Substituting the above representation into (5.1), we obtain the system of the compressible Navier-Stokes-Poisson type (NSP) as

$$\begin{cases} \partial_t n + \operatorname{div}_x m = 0, \\ \partial_t m + \nabla_x n + \sqrt{\frac{2}{3}} \nabla_x q - \nabla_x \Phi = \eta(\Delta_x m + \frac{1}{3} \nabla_x \operatorname{div}_x m) + n \nabla_x \Phi + R_1, \\ \partial_t q + \sqrt{\frac{2}{3}} \operatorname{div}_x m = \alpha \Delta_x q + \sqrt{\frac{2}{3}} \nabla_x \Phi \cdot m + R_2, \end{cases} \quad (5.3)$$

where the viscosity coefficients  $\eta$ ,  $\alpha$  and the remainder terms  $R_1$ ,  $R_2$  are defined by

$$\begin{aligned} \eta &= -(L^{-1}\mathbf{P}_1(v_1 \chi_2), v_1 \chi_2), \quad \alpha = -(L^{-1}\mathbf{P}_1(v_1 \chi_4), v_1 \chi_4), \\ R_1 &= - \int_{\mathbb{R}^3} v \cdot \nabla_x L^{-1}[\partial_t(\mathbf{P}_1 f) + \mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_1 f) - \mathbf{P}_1 G] v \sqrt{M} dv, \\ R_2 &= - \int_{\mathbb{R}^3} v \cdot \nabla_x L^{-1}[\partial_t(\mathbf{P}_1 f) + \mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_1 f) - \mathbf{P}_1 G] \chi_4 dv. \end{aligned}$$

Define the macroscopic temperature of the solution to the VPB system (1.5) as

$$\theta = q - \sqrt{\frac{1}{6}} m^2. \quad (5.4)$$

Then we obtain the system for density, momentum and temperature  $(n, m, \theta)$  as

$$\begin{cases} \partial_t n + \operatorname{div}_x m = 0, \\ \partial_t m + \nabla_x n + \sqrt{\frac{2}{3}} \nabla_x q - \nabla_x \Phi - \eta(\Delta_x m + \frac{1}{3} \nabla_x \operatorname{div}_x m) = n \nabla_x \Phi + R_1, \\ \partial_t \theta + \sqrt{\frac{2}{3}} \operatorname{div}_x m - \alpha \Delta_x \theta = R_2 + R_3, \end{cases} \quad (5.5)$$

where the reminder terms  $R_1, R_2$  are defined above and the term  $R_3$  is given by

$$\begin{aligned} R_3 &= \sqrt{\frac{2}{3}} \nabla_x (n + \sqrt{\frac{2}{3}} q) \cdot m - \sqrt{\frac{2}{3}} \eta (\Delta_x m + \frac{1}{3} \nabla_x \operatorname{div}_x m) \cdot m - \sqrt{\frac{2}{3}} n \nabla_x \Phi \cdot m \\ &\quad - \sqrt{\frac{2}{3}} m R_1 + \sqrt{\frac{1}{6}} \alpha \Delta_x (m^2). \end{aligned}$$

We first consider the linearized NSP system for  $(n, m, \theta)$ :

$$\begin{cases} \partial_t n + \operatorname{div}_x m = 0, \\ \partial_t m + \nabla_x n + \sqrt{\frac{2}{3}} \nabla_x q - \nabla_x \Phi - \eta(\Delta_x m + \frac{1}{3} \nabla_x \operatorname{div}_x m) = 0, \\ \partial_t \theta + \sqrt{\frac{2}{3}} \operatorname{div}_x m - \alpha \Delta_x \theta = 0. \end{cases} \quad (5.6)$$

with the initial data

$$(n, m, \theta)(x, 0) = (n_0, m_0, \theta_0)(x). \quad (5.7)$$

We have from [11, 23] the following optimal time convergence rates of the global solution to the Cauchy problem (5.6)–(5.7) below.

**Lemma 5.1** ([11, 23]). *Let  $U_0 = (n_0, m_0, \theta_0) \in H^N(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3)$ ,  $N \geq 4$  and denote  $(n, m, \theta)$  the global solution to the Cauchy problem (5.6)–(5.7). Then, it holds*

$$\begin{aligned}\|\partial_x^\alpha n(t)\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\left(\frac{3}{4}+\frac{|\alpha|}{2}\right)}(\|\partial_x^\alpha U_0\|_{L^2(\mathbb{R}_x^3)} + \|U_0\|_{L^1(\mathbb{R}_x^3)}), \\ \|\partial_x^\alpha m(t)\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\left(\frac{1}{4}+\frac{|\alpha|}{2}\right)}(\|\partial_x^\alpha U_0\|_{L^2(\mathbb{R}_x^3)} + \|U_0\|_{L^1(\mathbb{R}_x^3)}), \\ \|\partial_x^\alpha \theta(t)\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\left(\frac{3}{4}+\frac{|\alpha|}{2}\right)}(\|\partial_x^\alpha U_0\|_{L^2(\mathbb{R}_x^3)} + \|U_0\|_{L^1(\mathbb{R}_x^3)}).\end{aligned}$$

Moreover, if  $n_0 = 0$ , then

$$\begin{aligned}\|\partial_x^\alpha n(t)\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\left(\frac{5}{4}+\frac{|\alpha|}{2}\right)}(\|\partial_x^\alpha U_0\|_{L^2(\mathbb{R}_x^3)} + \|U_0\|_{L^1(\mathbb{R}_x^3)}), \\ \|\partial_x^\alpha m(t)\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\left(\frac{3}{4}+\frac{|\alpha|}{2}\right)}(\|\partial_x^\alpha U_0\|_{L^2(\mathbb{R}_x^3)} + \|U_0\|_{L^1(\mathbb{R}_x^3)}), \\ \|\partial_x^\alpha \theta(t)\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\left(\frac{3}{4}+\frac{|\alpha|}{2}\right)}(\|\partial_x^\alpha U_0\|_{L^2(\mathbb{R}_x^3)} + \|U_0\|_{L^1(\mathbb{R}_x^3)}).\end{aligned}$$

Then, we turn to deal with the nonlinear NSP type system (5.5) with initial data (5.7) in the vector form

$$\partial_t U = DU + H, \quad U(0) = U_0, \quad (5.8)$$

where  $U = (n, m, \theta)^T$ ,  $U_0 = (n_0, m_0, \theta_0)^T$ , the Fourier transform  $\hat{D}$  of the linear differential operator  $D$  is

$$\hat{D}(\xi) = \begin{pmatrix} 0, & -i\xi^T, & 0 \\ -i\xi(1 + \frac{1}{|\xi|^2}), & -\eta(|\xi|^2 I + \frac{1}{3}\xi \otimes \xi), & -i\sqrt{\frac{2}{3}}\xi \\ 0, & -i\sqrt{\frac{2}{3}}\xi^T, & -\alpha|\xi|^2 \end{pmatrix},$$

and the nonlinear term  $H$  is given by

$$H = (0, n\nabla_x \Phi + R_1, R_2 + R_3)^T.$$

The solution to the problem (5.8) can be represented by

$$U(t) = e^{tD}U_0 + \int_0^t e^{(t-s)D}Hds. \quad (5.9)$$

Then, we have the following theorem concerned with the time decay rate of the temperature under the same assumptions in Theorem 4.2.

**Theorem 5.2.** *Under the same assumptions in Theorem 4.2 that  $f_0 \in H^N \cap L^{2,1}$  for  $N \geq 4$  satisfying  $\|f_0\|_{H_w^N \cap L^{2,1}} \leq \delta_0$  with  $\delta_0 > 0$  small enough, let  $f$  be the global solution to the VPB system (1.5). Denote  $n_0 = (f_0, \chi_0)$ ,  $m_0 = (f_0, v\sqrt{M})$ , and  $\theta_0 = q_0 - \sqrt{\frac{1}{6}}m_0^2$  with  $q_0 = (f_0, \chi_4)$ , and let  $U = (n, m, \theta)^T$  be the corresponding macroscopic density, momentum and temperature of  $f$  and satisfy the problem (5.8) with  $U_0 = (n_0, m_0, \theta_0)^T$ . Then, it holds for  $t > 0$  and  $k = 0, 1$  that*

$$\|n(t)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{3}{4}}, \quad (5.10)$$

$$\|m(t)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{1}{4}}, \quad (5.11)$$

$$\|\theta(t)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{3}{4}+\varepsilon}, \quad (5.12)$$

where  $\varepsilon > 0$  is an any constant.

*Proof.* By Theorem 4.2, we only need to prove (5.12). Split the nonlinear term  $H$  into two parts

$$H = I + J, \quad I = (0, I_1, I_2), \quad J = (0, J_1, J_2),$$

where  $I$  is the linear part and  $J$  is the nonlinear part

$$I_1 = \int_{\mathbb{R}^3} v \cdot \nabla_x L^{-1}[\partial_t(\mathbf{P}_1 f) + \mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_1 f)]v\sqrt{M}dv,$$



$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^3} v \cdot \nabla_x L^{-1} [\partial_t (\mathbf{P}_1 f) + \mathbf{P}_1 (v \cdot \nabla_x \mathbf{P}_1 f)] \chi_4 dv, \\
J_1 &= \int_{\mathbb{R}^3} (v \cdot \nabla_x L^{-1} \mathbf{P}_1 G) v \sqrt{M} dv + n \nabla_x \Phi, \quad J_2 = \int_{\mathbb{R}^3} (v \cdot \nabla_x L^{-1} \mathbf{P}_1 G) \chi_4 dv + R_3.
\end{aligned}$$

Let us denote  $U = (U^0, U^1, U^2, U^3, U^4)$ . Since

$$e^{t\hat{D}(\xi)} \hat{U} = \sum_{j=-1}^3 e^{\lambda_j(|\xi|)t} (\hat{U}, \overline{\psi_j(\xi)})_{\xi} \psi_j(\xi), \quad (5.13)$$

where  $(U, V)_{\xi} = (U, V) + \frac{1}{|\xi|^2} U^0 \overline{V^0}$ , and  $\lambda_j(|\xi|)$ ,  $\psi_j(\xi)$ ,  $j = -1, 0, 1, 2, 3$ , are the eigenvalues and the corresponding normalized eigenfunctions of  $\hat{D}(\xi)$ , it follows that

$$e^{(t-s)\hat{D}(\xi)} \partial_s \hat{U} = -\partial_s \sum_{j=-1}^3 e^{\lambda_j(|\xi|)(t-s)} (\hat{U}, \overline{\psi_j(\xi)})_{\xi} \psi_j(\xi) + \sum_{j=-1}^3 e^{\lambda_j(|\xi|)(t-s)} \lambda_j(|\xi|) (\hat{U}, \overline{\psi_j(\xi)})_{\xi} \psi_j(\xi).$$

Then by [23], one has

$$(e^{(t-s)\hat{D}(\xi)} \partial_s \hat{U}, e_4) = -\partial_s (e^{(t-s)\hat{D}(\xi)} \hat{U}, e_4) + |\xi| T_8(t-s, \xi) \hat{U} 1_{|\xi| \leq r_0} + T_9(t-s, \xi) \hat{U} 1_{|\xi| > r_0},$$

where  $e_4 = (0, 0, 0, 0, 1)$ ,  $T_8(t, \xi)$  is the low frequency term satisfying  $|T_8(t, \xi) \hat{U}|^2 \leq C e^{-2a_1 |\xi|^2 t} |\hat{U}|^2$  and  $T_9(t, \xi)$  is the high frequency term satisfying  $|T_9(t, \xi) \hat{U}|^2 \leq C e^{-2a_1 t} |\hat{U}|^2$  for  $a_1 > 0$  some constant. This leads to

$$\begin{aligned}
&\int_0^t (e^{(t-s)\hat{D}(\xi)} \partial_s \hat{U}, e_4) ds \\
&= \int_0^t -\partial_s (e^{(t-s)\hat{D}(\xi)} \hat{U}, e_4) ds + \int_0^t (|\xi| T_8(t-s, \xi) \hat{U} 1_{|\xi| \leq r_0} + T_9(t-s, \xi) \hat{U} 1_{|\xi| > r_0}) ds \\
&= (e^{t\hat{D}(\xi)} \hat{U}_0, e_4) - \hat{U}^4(t) + \int_0^t (|\xi| T_8(t-s, \xi) \hat{U} 1_{|\xi| \leq r_0} + T_9(t-s, \xi) \hat{U} 1_{|\xi| > r_0}) ds.
\end{aligned} \quad (5.14)$$

Note that

$$\theta(t) = (e^{tD} U_0, e_4) + \int_0^t (e^{(t-s)D} H, e_4) ds,$$

we obtain by Lemma 5.1 that

$$\|\theta(t)\|_{L^2(\mathbb{R}_x^3)} \leq C(1+t)^{-\frac{3}{4}} + \left\| \int_0^t (e^{(t-s)D} I, e_4) ds \right\|_{L^2(\mathbb{R}_x^3)} + \left\| \int_0^t (e^{(t-s)D} J, e_4) ds \right\|_{L^2(\mathbb{R}_x^3)}.$$

Then we estimate the terms on the right hand side of above. By (5.14) and Lemma 5.1, the linear part is bounded by

$$\begin{aligned}
\left\| \int_0^t (e^{(t-s)D} I, e_4) ds \right\|_{L^2(\mathbb{R}_x^3)} &\leq C(1+t)^{-\frac{3}{4}} \|\mathbf{P}_1 f_0\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \\
&\quad + \int_0^t (1+t-s)^{-1} \|\mathbf{P}_1 f\|_{L^2(\mathbb{R}_{x,v}^6)} ds \\
&\leq C(1+t)^{-\frac{3}{4}} + \int_0^t (1+t-s)^{-1} (1+s)^{-\frac{3}{4}} ds \leq C(1+t)^{-\frac{3}{4}+\varepsilon},
\end{aligned} \quad (5.15)$$

and the nonlinear part is bounded by

$$\begin{aligned}
\left\| \int_0^t (e^{(t-s)D} J, e_4) ds \right\|_{L^2(\mathbb{R}_x^3)} &\leq C \int_0^t (1+t-s)^{-\frac{3}{4}} (\|R_3\|_{L^2(\mathbb{R}_x^3)} + \|R_3\|_{L^1(\mathbb{R}_x^3)}) ds \\
&\quad + C \int_0^t (1+t-s)^{-\frac{3}{4}} (\|\nabla_x G\|_{L^2(\mathbb{R}_{x,v}^6)} + \|\nabla_x G\|_{L^{2,1}}) ds
\end{aligned}$$

$$\leq C \int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-1} ds \leq C(1+t)^{-\frac{3}{4}+\varepsilon}, \quad (5.16)$$

where  $\varepsilon > 0$  is a small but fixed constant and

$$\|R_3\|_{L^2(\mathbb{R}_x^3)} + \|R_3\|_{L^1(\mathbb{R}_x^3)} \leq C(1+t)^{-1}.$$

The combination of (5.15)–(5.16) leads to (5.12). This completes the proof of the Lemma.  $\square$

**Remark 5.3.** By Theorem 5.2 the time decay rate of the temperature  $\theta$  to (5.8) is  $(1+t)^{-3/4+\varepsilon}$ , which is slower than the optimal algebraic rate  $(1+t)^{-3/4}$  of temperature established in [23] for the macroscopic compressible non-isentropic Navier-Stokes-Poisson equations. The main reason is the coupling of the temperature equation with the microscopic part  $\partial_t(\mathbf{P}_1 f) + \mathbf{P}_1(v \cdot \nabla_x \mathbf{P}_1 f)$  in (5.5) and (5.8), which reduces the time convergence rate of the temperature to its equilibrium state according to the optimal decay rates of microscopic part of  $F$  shown by Theorem 4.2. Indeed, if the nonlinear terms  $R_1$  and  $R_2$  are removed from the equations (5.5), the system (5.5) becomes the macroscopic compressible non-isentropic Navier-Stokes-Poisson equations for the density, momentum and temperature, and then one can prove the same optimal time decay rates of the global solution to (5.5) and (5.7) as those in [23], in particular, the temperature decays like  $(1+t)^{-3/4}$ .

## 5.2 Comparison with Boltzmann equation

For the Cauchy problem of the Boltzmann equation

$$\begin{cases} F_t + v \cdot \nabla_x F = \mathcal{Q}(F, F), \\ F(x, v, 0) = F_0(x, v), \end{cases} \quad (5.17)$$

which after taken the decomposition  $F = M + \sqrt{M}f$ , it can be re-written as

$$\begin{cases} f_t = Ef + \Gamma(f, f), \\ f(x, v, 0) = f_0(x, v) =: M^{-\frac{1}{2}}(F_0 - M)(x, v), \end{cases} \quad (5.18)$$

where the operator  $E = L - (v \cdot \nabla_x)$  and the operators  $L$  and  $\Gamma$  are defined by (1.8) and (1.9) respectively.

Based on the well-known result on the spectrum obtained by [7], we can show the following theorem on the optimal convergence decay rates.

**Theorem 5.4** ([24]). Suppose that  $f_0 \in L^2(\mathbb{R}_v^3; H^N(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3))$  with  $N \geq 4$ ,  $\|\sqrt{v} f_0\|_{L^2(\mathbb{R}_v^3; H^N(\mathbb{R}_x^3))} + \|f_0\|_{L^{2,1}} \leq \delta_0$  with  $\delta_0 > 0$  being small enough. Assume that there exist two positive constants  $d_0 > 0$  and  $d_1 > 0$  such that  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \sqrt{M})| \geq d_0$ ,  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_0, \sqrt{M})|$  and  $(\hat{f}_0, v\sqrt{M}) = 0$  for  $|\xi| \leq r_0$ . Then, the global solution  $f$  to the Cauchy problem (5.18) satisfies for time  $t > 0$  that

$$\begin{aligned} \|\partial_x^k(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} &\leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 0, 1, 2, 3, 4, \\ \|\partial_x^k \mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} &\leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \\ \|\sqrt{v} \mathbf{P}_1 f\|_{L^2(\mathbb{R}_v^3; H^N(\mathbb{R}_x^3))} + \|\nabla_x \mathbf{P}_0 f\|_{L^2(\mathbb{R}_v^3; H^{N-1}(\mathbb{R}_x^3))} &\leq C\delta_0(1+t)^{-\frac{5}{4}}, \end{aligned}$$

for  $k = 0, 1$ , and for time  $t > 0$  large enough that

$$\begin{aligned} C_1\delta_0(1+t)^{-\frac{3}{4}} &\leq \|(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C_2\delta_0(1+t)^{-\frac{3}{4}}, \quad j = 0, 1, 2, 3, 4, \\ C_1\delta_0(1+t)^{-\frac{5}{4}} &\leq \|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C_2\delta_0(1+t)^{-\frac{5}{4}}, \\ C_1\delta_0(1+t)^{-\frac{3}{4}} &\leq \|f(t)\|_{L^2(\mathbb{R}_v^3; H^N(\mathbb{R}_x^3))} \leq C_2\delta_0(1+t)^{-\frac{3}{4}}, \end{aligned}$$

with two positive constants  $C_2 \geq C_1$ . In addition, the same upper and lower bounds of the decay rates do not change for the global solution  $f$  if it further holds  $(f_0, \chi_0) = 0$ .

If  $\mathbf{P}_0 f_0 = 0$ ,  $\inf_{|\xi| \leq r_0} |(\hat{f}_0, L^{-1} \mathbf{P}_1(v \cdot \omega)^2 \sqrt{M})| \geq d_0$  and  $(\hat{f}_0, L^{-1} \mathbf{P}_1(v \cdot \omega) \chi_4) = 0$  for  $|\xi| \leq r_0$ , then

$$\begin{aligned} \|\partial_x^k(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} &\leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 0, 1, 2, 3, 4, \\ \|\partial_x^k \mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} &\leq C\delta_0(1+t)^{-\frac{7}{4}-\frac{k}{2}}, \\ \|\sqrt{\nu} \mathbf{P}_1 f\|_{L^2(\mathbb{R}_v^3; H^N(\mathbb{R}_x^3))} + \|\nabla_x \mathbf{P}_0 f\|_{L^2(\mathbb{R}_v^3, H^{N-1}(\mathbb{R}_x^3))} &\leq C\delta_0(1+t)^{-\frac{7}{4}}, \end{aligned}$$

for  $k = 0, 1$ , and for time  $t > 0$  large enough that

$$\begin{aligned} C_1\delta_0(1+t)^{-\frac{5}{4}} &\leq \|(f(t), \chi_j)\|_{L^2(\mathbb{R}_x^3)} \leq C_2\delta_0(1+t)^{-\frac{5}{4}}, \quad j = 0, 1, 2, 3, 4, \\ C_1\delta_0(1+t)^{-\frac{7}{4}} &\leq \|\mathbf{P}_1 f(t)\|_{L^2(\mathbb{R}_{x,v}^6)} \leq C_2\delta_0(1+t)^{-\frac{7}{4}}, \\ C_1\delta_0(1+t)^{-\frac{5}{4}} &\leq \|f(t)\|_{L^2(\mathbb{R}_v^3, H^N(\mathbb{R}_x^3))} \leq C_2\delta_0(1+t)^{-\frac{5}{4}}, \end{aligned}$$

with two positive constants  $C_2 \geq C_1$ .

## 6 Appendix

Let us list the following results on the semigroup theory (cf. [16]) for the easy reference by the readers. Let  $H$  be a Hilbert space with the inner product denoted by  $(\cdot, \cdot)$ .

**Definition 6.1.** A linear operator  $A$  is dissipative if  $\operatorname{Re}(Af, f) \leq 0$  for every  $f \in D(A) \subset H$ .

**Lemma 6.2.** Let  $A$  be a densely defined closed linear operator on  $H$ . If both  $A$  and its adjoint operator  $A^*$  are dissipative, then  $A$  is the infinitesimal generator of a  $C_0$ -semigroup on  $H$ .

The following basic results are from [16].

**Lemma 6.3** (Stone). The operator  $A$  is the infinitesimal generator of a continuous unitary group on a Hilbert space  $H$  if and only if the operator  $iA$  is self-adjoint.

**Lemma 6.4.** Let  $A$  be the infinitesimal generator of a  $C_0$ -semigroup  $T(t)$  satisfying  $\|T(t)\| \leq Me^{\kappa t}$ . Then, it holds for  $f \in D(A^2)$  and  $\sigma > \max(0, \kappa)$  that

$$T(t)f = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} (\lambda - A)^{-1} f d\lambda. \quad (6.19)$$

**Lemma 6.5.** Let  $A$  be the infinitesimal generator of the  $C_0$  semigroup  $T(t)$ . If  $D(A^n)$  is the domain of  $A^n$ , then  $\cap_{n=1}^{\infty} D(A^n)$  is dense in  $X$ .

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